Twisted Alexander polynomials of 2-bridge knots associated to metacyclic representations

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ABSTRACT

Let $p = 2n+1$ be a prime and D_p a dihedral group of order $2p$. Let $\widehat{\rho} : G(K) \to D_p \to GL(p, \mathbb{Z})$ be a non-abelian representation of the knot group $G(K)$ of a knot K in 3-sphere. Let $\widetilde{\Delta}_{\widehat{\rho},K}(t)$ be the twisted Alexander polynomial of K associated to $\widehat{\rho}$. Then we prove that for any 2-bridge knot $K(r)$ in $H(p)$, $\widetilde{\Delta}_{\widehat{\rho},K}(t)$ is of the form $\left\{\dfrac{\Delta_{K(r)}(t)}{1-t}\right\} f(t)f(-t)$ for some integer polynomial $f(t)$, where $H(p)$ is the set of 2-bridge knots $K(r), 0 < r < 1$, such that $G(K(r))$ is mapped onto a non-trivial free product $\mathbb{Z}/2 * \mathbb{Z}/p$. Further, it is proved that $f(t) \equiv \left\{\dfrac{\Delta_K(t)}{1+t}\right\}^n$ (mod p), where $\Delta_K(t)$ is the Alexander polynomial of K. Later we discuss the twisted Alexander polynomial associated to the general metacyclic representation.

Keywords: 2-bridge knot, twisted Alexander polynomial, dihedral representation, metacyclic representation.

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1. Introduction

In the previous paper [7], we studied the parabolic representation of the group of a 2-bridge knot and showed some properties of its twisted Alexander polynomial. In this paper, we consider a metacyclic representations of the knot group.

Let $G(m, p|k)$ be a (non-abelian) semi-direct product of two cyclic groups $\mathbb{Z}/m$ and $\mathbb{Z}/p$, p an odd prime, with the following presentation:

$$G(m, p|k) = \langle s, a | s^m = a^p = 1, sas^{-1} = a^k \rangle, \tag{1.1}$$

where k is a primitive m-th root of 1 (mod p), i.e. $k^m \equiv 1$ (mod p), but $k^q \not\equiv 1$ (mod p) for any $q, 0 < q < m$ and $k \neq 0, 1$.

If $k=-1$, then $m=2$ and hence $G(2,p|-1)$ is a dihedral group D_p. Since k is a primitive m-th root of 1 (mod p), $G(m,p|k)$ is imbedded in the symmetric group S_p and hence in $GL(p,\mathbb{Z})$ via permutation matrices.

Now suppose that the knot group $G(K)$ of a knot K is mapped onto $G(m,p|k)$ for some m,p and k. Then, we have a representation $f:\ G(K)\to G(m,p|k)\to GL(p,\mathbb{Z})$ and the twisted Alexander polynomial $\widetilde{\Delta}_{f,K}(t)$ associated to f is defined [11] [15] [10]. One of our objectives is to characterize these twisted Alexander polynomials. In fact, we propose the following conjecture.

Conjecture A. *$\widetilde{\Delta}_{f,K}(t)=\left\{\dfrac{\Delta_K(t)}{1-t}\right\}F(t)$, where $\Delta_K(t)$ is the Alexander polynomial of K and $F(t)$ is an integer polynomial in t^m.*

First we study the case $k=-1$, dihedral representations of the knot group. Let D_p be a dihedral group of order $2p$, where $p=2n+1$ and p is a prime. Then the knot group $G(K)$ of a knot K is mapped onto D_p if and only if $\Delta_K(-1)\equiv 0$ (mod p) [2], [5]. Therefore, if $\Delta_K(-1)\neq\pm1$, $G(K)$ has at least one representation on a certain dihedral group D_p. For these cases, we can make Conjecture A slightly sharper:

Conjecture B. *Let $\widehat{\rho}:\ G(K)\to D_p\to GL(p,\mathbb{Z})$ be a non-abelian representation of the knot group $G(K)$ of a knot K and let $\widetilde{\Delta}_{\widehat{\rho},K}(t)$ be the twisted Alexander polynomial of K associated to $\widehat{\rho}$. Then*

$$\widetilde{\Delta}_{\widehat{\rho},K}(t)=\left\{\frac{\Delta_K(t)}{1-t}\right\}f(t)f(-t), \tag{1.2}$$

where $f(t)$ is an integer polynomial and further,

$$f(t)\equiv\left\{\frac{\Delta_K(t)}{1+t}\right\}^n \pmod{p} \tag{1.3}$$

We should note that $(1+t)^2$ divides $\Delta_K(t)$ (mod p) if and only if $\Delta_K(-1)\equiv 0$ (mod p).

The main purpose of this paper is to prove (1.2) for a 2-bridge knot $K(r)$ in $H(p)$, p a prime, and (1.3) for a 2-bridge knot with $\Delta_K(-1)\equiv 0$ (mod p). (See Theorem 2.2.) Here $H(p)$ is the set of 2-bridge knots $K(r), 0<r<1$, such that $G(K(r))$ is mapped onto a free product $\mathbb{Z}/2*\mathbb{Z}/p$. We note that knots in $H(p)$ have been studied extensively in [4] and [13].

A proof of the main theorem (Theorem 2.2) is given in Section 2 through Section 7. Since this paper is a sequel of [7], we occasionally skip some details if the argument used in [7] also works in this paper.

In Section 8, we consider another type of metacyclic groups, denoted by $N(q,p)$. $N(q,p)$ is a semi-direct product of two cyclic groups, $\mathbb{Z}/2q$ and $\mathbb{Z}/p$ defined by

$$N(q,p)=\langle s,a|s^{2q}=a^p=1, sas^{-1}=a^{-1}\rangle, \tag{1.4}$$

where $q \geq 1$ and p is an odd prime and $\gcd(q, p) = 1$. We note that $N(1, p) = D_p$ and $N(2, p)$ is called a binary dihedral group.

Let $\widetilde{\nu} : \ G(K) \longrightarrow N(q, p) \longrightarrow GL(2pq, \mathbb{Z})$ be a representation of $G(K)$. (For details, see Section 8.) Then we show that for a 2-bridge knot $K(r)$, the twisted Alexander polynomial $\widetilde{\Delta}_{\widetilde{\nu},K(r)}(t)$ associated to $\widetilde{\nu}$ is completely determined by the Alexander polynomial $\Delta_{K(r)}(t)$ and the twisted Alexander polynomial $\widetilde{\Delta}_{\widehat{\rho},K(r)}(t)$ associated to $\widehat{\rho}$. (Proposition 8.5)

In Section 9, we give examples that illustrate our main theorem and Proposition 8.5. It is interesting to observe that $\widetilde{\Delta}_{\widetilde{\nu},K(r)}(t)$ is an integer polynomial in t^{2q}. In Section 10, we briefly discuss general $G(m, p|k)$-representations of the knot group and give several examples, one of which is not a 2-bridge knot, that support Conjecture A. In Section 11, we prove Proposition 2.1 and Lemma 5.2 that plays a key role in our proof of the main theorem.

Finally, for convenience, we draw a diagram below consisting of homomorphisms that connect various groups and rings.

$$
\begin{array}{ccccccccc}
 & & GL(p,\mathbb{Z}) & & GL(2n,\mathbb{Z}) & & \\
 & & \pi \uparrow & \nearrow \pi_0 & \uparrow \gamma & & \\
G(K) & \underset{\rho}{\longrightarrow} & D_p & \underset{\xi}{\longrightarrow} & GL(2,\mathbb{C}) & & \\
\downarrow & & \downarrow & & & & \\
\mathbb{Z}G(K) & \longrightarrow & \mathbb{Z}D_p & \underset{\zeta}{\longrightarrow} & \widetilde{A}(\omega) & & M_{2n,2n}(\mathbb{Z}[t^{\pm 1}]) \\
 & \rho^* \searrow & \downarrow & & \downarrow & & \uparrow \gamma^* \\
 & & \mathbb{Z}D_p[t^{\pm 1}] & \underset{\zeta^*}{\longrightarrow} & \widetilde{A}(\omega)[t^{\pm 1}] & \underset{\xi^*}{\longrightarrow} & M_{2,2}\big((\mathbb{Z}[\omega])[t^{\pm 1}]\big)
\end{array}
$$

Here, $\tau = \rho \circ \xi, \widehat{\rho} = \rho \circ \pi, \rho_0 = \rho \circ \pi_0, \eta = \xi \circ \gamma, \Phi^* = \rho^* \circ \zeta^* \circ \xi^*$ and $\nu = \rho \circ \xi \circ \gamma$. Unmarked arrows indicate natural extensions of homomorphisms.

2. Dihedral representations and statement of the main theorem

We begin with a precise formulation of representations. Let $p = 2n + 1$ and D_p be a dihedral group of order $2p$ with a presentation: $D_p = \langle x, y | x^2 = y^2 = (xy)^p = 1 \rangle$. As is well known, D_p can be faithfully represented in $GL(p, \mathbb{Z})$ by the map π defined by:

$$
x \mapsto \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & & \iddots & & \vdots \\
\vdots & \vdots & & \iddots & & \vdots & \vdots \\
\vdots & & \iddots & & \vdots & \vdots & \vdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\quad
y \mapsto \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & & \iddots & & \vdots \\
\vdots & \vdots & & \iddots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & 0 & 0 & 0
\end{bmatrix}
$$

However, π is reducible. In fact, π is equivalent to $id * \pi_0$, where

$$\pi_0 : x \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & & \iddots & & \vdots \\ \vdots & & \iddots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \; y \mapsto \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & & \iddots & & \vdots \\ \vdots & \vdots & & \iddots & & \vdots & \vdots \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{2.1}$$

For convenience, π_0 is called the *irreducible representation* of D_p (of degree $p-1=2n$).

Now let $K(r), 0 < r < 1, r = \frac{\beta}{\alpha}$ and $\gcd(\alpha, \beta) = 1$, be a 2-bridge knot and consider a Wirtinger presentation of the group $G(K(r))$:

$$\begin{aligned} & G(K(r)) = \langle x, y | R \rangle, \text{ where} \\ & R = WxW^{-1}y^{-1}, W = x^{\epsilon_1} y^{\epsilon_2} \cdots x^{\epsilon_{\alpha-2}} y^{\epsilon_{\alpha-1}} \text{ and} \\ & \epsilon_j = \pm 1 \text{ for } 1 \le j \le \alpha - 1. \end{aligned} \tag{2.2}$$

Suppose p be a prime. If $\alpha \equiv 0 \pmod p$, then a mapping

$$\rho : x \mapsto x \text{ and } y \mapsto y \tag{2.3}$$

defines a surjection from $G(K(r))$ to D_p.

Therefore $\rho_0 = \rho \circ \pi_0$ defines a representation of $G(K(r))$ into $GL(2n, \mathbb{Z})$ and we can define the twisted Alexander polynomial $\widetilde{\Delta}_{\rho_0, K(r)}(t)$ associated to ρ_0. Since $\pi = id * \pi_0$, the twisted Alexander polynomial associated to $\widehat{\rho} = \rho \circ \pi$ is given by $\left[\dfrac{\Delta_{K(r)}(t)}{1-t}\right] \widetilde{\Delta}_{\rho_0, K(r)}(t)$ and hence (1.2) becomes

$$\widetilde{\Delta}_{\rho_0, K(r)}(t) = f(t) f(-t). \tag{2.4}$$

Now there is another representation of D_p in $GL(2, \mathbb{C})$. To be more precise, consider $\xi : D_p \to GL(2, \mathbb{C})$ given by

$$\xi(x) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \xi(y) = \begin{bmatrix} -1 & 0 \\ \omega & 1 \end{bmatrix}, \tag{2.5}$$

where $\omega \in \mathbb{C}$ is determined as follows.

First we set $\xi(x) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\xi(y) = \begin{bmatrix} -1 & 0 \\ z & 1 \end{bmatrix}$, and write $\xi((xy)^k) = \begin{bmatrix} a_k(z) & b_k(z) \\ c_k(z) & d_k(z) \end{bmatrix}$. Since $\xi(xy) = \begin{bmatrix} 1+z & 1 \\ z & 1 \end{bmatrix}$, we see that a_k, b_k, c_k and d_k are exactly the same polynomials found in [7, (4.1)]. Further, as is mentioned in [7], $a_n(z)$ and $b_n(z)$ are given as follows: [7, Propositions 10.2 and 2.4]:

$$a_n(z) = \sum_{k=0}^{n} \binom{n+k}{2k} z^k \text{ and } b_n(z) = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} z^k. \tag{2.6}$$

Since $(xy)^{2n+1} = 1$, we have $(xy)^n x = y(xy)^n$ and hence, a simple calculation shows that $\xi((xy)^n x) = \xi(y(xy)^n)$ yields $a_n(z)+2b_n(z) = 0$. Therefore, the number ω we are looking for is a root of $\theta_n(z) = a_n(z) + 2b_n(z)$. Write $\theta_n(z) = c_0{}^{(n)} + c_1{}^{(n)}z + \cdots + c_{n-1}{}^{(n)}z^{n-1} + c_n{}^{(n)}z^n$. Then we see

$$c_k{}^{(n)} = \binom{n+k}{2k} + 2\binom{n+k}{2k+1} = \frac{2n+1}{2k+1}\binom{n+k}{n-k}. \tag{2.7}$$

If $p = 2n+1$ is prime, then, for $0 \le k \le n-1$, $c_k{}^{(n)} \equiv 0 \pmod p$, but $c_0{}^{(n)} = p$ and $c_n{}^{(n)} = 1$. Therefore, by Eisenstein's criterion, $\theta_n(z)$ is irreducible and it is the minimal polynomial of ω.

Let C_n be the companion matrix of $\theta_n(z)$. By substituting C_n for ω, we have a homomorphism $\gamma : GL(2,\mathbb{C}) \to GL(2n,\mathbb{Z})$, namely, $\gamma(1) = E_n$ and $\gamma(\omega) = C_n$, where E_n is the identity matrix, and hence we obtain another representation $\eta = \xi \circ \gamma : D_p \to GL(2n,\mathbb{Z})$.

The following proposition is likely known, but since we are unable to find a reference, we prove it in Section 11.

Proposition 2.1. *Two representations π_0 and η are equivalent. In other words, there is a matrix $U_n \in GL(2n,\mathbb{Z})$ such that*

$$U_n\pi_0(x)U_n{}^{-1} = \eta(x) \text{ and } U_n\pi_0(y)U_n{}^{-1} = \eta(y). \tag{2.8}$$

Let $K(r)$ be a 2-bridge knot in $H(p)$. Then $\tau = \rho \circ \xi : G(K(r)) \to D_p \to GL(2,\mathbb{C})$ defines a representation of $G(K(r))$ and let $\widetilde{\Delta}_{\tau,K(r)}(t|\omega)$ be the twisted Alexander polynomial associated to τ. Sometimes, we use the notation $\widetilde{\Delta}_{\tau,K(r)}(t|\omega)$ to emphasize that the polynomial involves ω. Let $\omega_1, \omega_2, \cdots, \omega_n$ be all the roots of $\theta_n(t)$. Since $\theta_n(t)$ is irreducible, the total τ-twisted Alexander polynomial $D_{\tau,K(r)}(t)$ defined in [14] is given by

$$D_{\tau,K(r)}(t) = \prod_{j=1}^{n} \widetilde{\Delta}_{\tau,K(r)}(t|\omega_j). \tag{2.9}$$

It is known that the polynomial $D_{\tau,K(r)}(t)$ is rewritten as

$$D_{\tau,K(r)}(t) = \det[\widetilde{\Delta}_{\tau,K(r)}(t|\omega)]^{\gamma}. \tag{2.10}$$

By (2.5), we see that $D_{\tau,K(r)}(t)$ is exactly the twisted Alexander polynomial of $K(r)$ associated to $\nu = \rho \circ \eta : G(K) \to GL(2n,\mathbb{Z})$. Since, by Proposition 2.1, π_0 and η are equivalent, ρ_0 and ν are equivalent, and hence $\widetilde{\Delta}_{\rho_0,K(r)}(t) = D_{\tau,K(r)}(t)$.

Conjecture A now becomes the following theorem under our assumptions that will be proven in Sections 5-7.

Theorem 2.2. *If a 2-bridge knot $K(r)$ is in $H(p)$, then*

$$D_{\tau,K(r)}(t) = f(t)f(-t) \tag{2.11}$$

for some integer polynomial $f(t)$, and further, for any 2-bridge knot $K(r)$ with $\Delta_{K(r)}(-1) \equiv 0$ (mod p),

$$\begin{aligned}&(1)\ D_{\tau,K(r)}(t) \equiv f(t)f(-t) \pmod p \text{ and}\\ &(2)\ f(t) \equiv \left\{\frac{\Delta_K(t)}{1+t}\right\}^n \pmod p,\end{aligned} \tag{2.12}$$

where $\Delta_{K(r)}(t)$ is the Alexander polynomial of $K(r)$.

We note that $\Delta_{K(r)}(t)$ is divisible by $1+t$ in $(\mathbb{Z}/p)[t^{\pm 1}]$.

Remark 2.3. If $n = 1$, i.e., $p = 3$, $\theta_1(z) = z + 3$, and hence $\omega = -3$. Therefore, γ is an identity homomorphism and $\widetilde{\Delta}_{\rho_0,K(r)}(t) = D_{\tau,K(r)}(t)$.

3. Basic formulas

In this section, we list various formulas involving a_k, b_k, c_k and d_k which will be used throughout this paper. Most of these materials are collected from Section 4 in [7].

For simplicity, let $\xi(x) = X = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\xi(y) = Y = \begin{bmatrix} -1 & 0 \\ \omega & 1 \end{bmatrix}$, where ω is a root of $\theta_n(z)$.

First we list several formulas which are similar to [7, Proposition 4.2]

Proposition 3.1. *As before, write* $(XY)^k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$.

$$\begin{aligned}
&(I)\ a_0 = d_0 = 1 \text{ and } b_0 = c_0 = 0.\\
&(II)\ a_1 = 1+\omega, b_1 = 1, c_1 = \omega \text{ and } d_1 = 1.\\
&(III)\ (i)\ \text{For } k \geq 2,\\
&\quad (1)\ a_k = (2+\omega)a_{k-1} - a_{k-2},\\
&\quad (2)\ \omega b_k = (1+\omega)a_{k-1} - a_{k-2},\\
&\qquad (ii)\ \text{For } k \geq 1,\\
&\quad (3)\ \omega b_k = a_k - a_{k-1},\\
&\quad (4)\ \omega b_k = c_k,\\
&\quad (5)\ a_k = \omega b_k + d_k,\\
&\quad (6)\ d_k = a_{k-1},\\
&\quad (7)\ b_k = b_{k-1} + a_{k-1},\\
&\quad (8)\ c_k + d_k = a_k,\\
&\quad (9)\ a_0 + a_1 + \cdots + a_{k-1} = b_k.
\end{aligned} \tag{3.1}$$

Since a proof of Proposition 3.1 is exactly the same as that of Proposition 4.2 in [7], we omit the details.

Next three propositions are different from the corresponding proposition [7, Proposition 4.4], since they depend on defining relations of D_p.

Proposition 3.2. *Let $p = 2n + 1$.*

$$\begin{aligned} &(1)\ \textit{For } 0 \leq k \leq 2n, a_k = a_{2n-k} \textit{ and } a_{2n+1} = a_0. \\ &(2)\ \textit{For } 0 \leq k \leq 2n, b_k = -b_{p-k} \textit{ and } b_p = 0. \end{aligned} \tag{3.2}$$

Proof. Since $(XY)^k = (YX)^{p-k} = Y(XY)^{p-k}Y$, we have $\begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} = \begin{bmatrix} a_{p-k} - \omega b_{p-k} & -b_{p-k} \\ -\omega a_{p-k} - c_{p-k} + \omega^2 b_{p-k} + \omega d_{p-k} & \omega b_{p-k} + d_{p-k} \end{bmatrix}$ and hence $a_k = a_{p-k} - \omega b_{p-k}$ and $b_k = -b_{p-k}$ which proves (2). Further, $a_k = a_{p-k} - \omega b_{p-k} = a_{p-k} + \omega b_k$ and thus, $a_{p-k} = a_{k-1}$ by (3.1)(III)(3). This proves (1). Finally, it is obvious that $a_p = a_0$. □

Proposition 3.3. *Let $p = 2n + 1$. Then we have the following*

$$\begin{aligned} &(1)\ a_0 + a_1 + \cdots + a_{2n} = 0, \\ &(2)\ b_1 + b_2 + \cdots + b_{2n} = 0, \\ &(3)\ d_0 + d_1 + \cdots + d_{2n} = 0, \\ &(4)\ a_n + 2b_n = 0. \\ &(5)\ \textit{If } k \equiv \ell \pmod{p}, \textit{ then } a_k = a_\ell, b_k = b_\ell, c_k = c_\ell \textit{ and } d_k = d_\ell. \end{aligned} \tag{3.3}$$

Proof. First, we see that $(XY)^n X = Y(XY)^n$ implies $\begin{bmatrix} -a_n & a_n + b_n \\ -c_n & c_n + d_n \end{bmatrix} = \begin{bmatrix} -a_n & -b_n \\ \omega a_n + c_n & \omega b_n + d_n \end{bmatrix}$, and hence $a_n + b_n = -b_n$ that proves (4). (5) is immediate, since $(XY)^p = 1$. (1) follows from (3.1)(III)(9), since $a_0 + a_1 + \cdots + a_{2n} = b_{2n+1} = 0$. To show (2), use (3.1)(III)(3). Since $b_0 = 0$, we see $\omega(b_1 + b_2 + \cdots + b_{2n}) = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_{2n-1} - a_{2n-2}) + (a_{2n} - a_{2n-1}) = a_{2n} - a_0 = 0$, by (3.2)(1). (3) follows from (3.1)(III)(6), since $d_0 = 1 = a_0 = a_{2n}$ and $d_0 + d_1 + \cdots + d_{2n} = 1 + a_0 + a_1 + \cdots + a_{2n-1} = a_0 + a_1 + \cdots + a_{2n-1} + a_{2n} = 0$. □

Now we define an algebra $\widetilde{A}(\omega)$ using the group ring $\mathbb{Z}D_p$. Consider the linear extension $\widehat{\xi}$ of $\xi : \mathbb{Z}D_p \to M_{2,2}(\mathbb{Z}[\omega])$ given by $\widehat{\xi}(x) = X$ and $\widehat{\xi}(y) = Y$, where $M_{k,k}(R)$ denotes the ring of $k \times k$ matrices over a commutative ring R. Let $\widehat{\xi}^{-1}(0)$ be the kernel of $\widehat{\xi}$. Then $\widetilde{A}(\omega) = \mathbb{Z}D_p/\widehat{\xi}^{-1}(0)$ is a non-commutative $\mathbb{Z}[\omega]$-algebra. Some elements of $\widehat{\xi}^{-1}(0)$ can be found in Proposition 3.4 below.

We define $\zeta : \mathbb{Z}D_p \to \widetilde{A}(\omega)$ to be the natural projection.

Proposition 3.4. *In $\widetilde{A}(\omega)$, the following formulas hold, where 1 denotes the identity of $\widetilde{A}(\omega)$.*

$$\textit{For } 1 \leq k \leq n, (xy)^k + (yx)^k = (a_{k-1} + a_k)1. \tag{3.4}$$

$$\begin{aligned} &(1)\ \textit{For } 1 \le k \le n-1, (xy)^k x + y(xy)^k = a_k(x+y),\\ &(2)\ (xy)^n x = y(xy)^n = \frac{a_n}{2}(x+y) = -b_n(x+y). \end{aligned} \qquad (3.5)$$

Proof. To prove (3.4), it suffices to show that $(XY)^k + (YX)^k = (a_{k-1}+a_k)E_n$. In fact, for $1 \le k \le n$,

$$(XY)^k + (YX)^k = (XY)^k + (XY)^{p-k} = \begin{bmatrix} a_k + a_{p-k} & b_k + b_{p-k} \\ c_k + c_{p-k} & d_k + d_{p-k} \end{bmatrix}.$$

Since $a_k + a_{p-k} = a_k + a_{k-1}$ by (3.2)(1), $b_k + b_{p-k} = 0$ by (3.2)(2), $c_k + c_{p-k} = \omega(b_k + b_{p-k}) = 0$ and $d_k + d_{p-k} = a_{k-1} + a_{2n-k} = a_{k-1} + a_k$ by (3.1)(6) and (3.2)(1), (3.4) follows immediately. Next, for $1 \le k \le n-1$, $(XY)^k X + Y(XY)^k = \begin{bmatrix} -2a_k & a_k \\ \omega a_k & 2(c_k + d_k) \end{bmatrix} = a_k(X+Y)$, which proves (3.5)(1). Finally, (3.5)(2) follows, since $(xy)^n x = y(xy)^n$ and $a_n = -2b_n$. □

4. Polynomials over $\widetilde{A}(\omega)$

In this section, as the first step toward a proof of Theorem 2.2, we introduce one of our key concepts in this paper.

Definition 4.1. Let $\varphi(t)$ be a polynomial on $t^{\pm 1}$ with coefficients in the non-commutative algebra $\widetilde{A}(\omega)$. We say $\varphi(t)$ is *split* if $\varphi(t)$ is of the form:
$\varphi(t) = \sum_j \alpha_j t^{2j} + \sum_k \beta_k (x+y) t^{2k+1}$, where $\alpha_j, \beta_k \in \mathbb{Z}[\omega]$. The set of split polynomials is denoted by $S(t)$. For example, $\varphi(t) = 1 + t^2, (x+y)t$ are split.

First we show that $S(t)$ is a commutative ring.

Proposition 4.2. *If $\varphi(t)$ and $\varphi'(t)$ are split, so are $\varphi(t) + \varphi'(t)$ and $\varphi(t)\varphi'(t)$.*

Proof. Let $\varphi(t) = \sum_j \alpha_j t^{2j} + \sum_k \beta_k (x+y) t^{2k+1}$ and $\varphi'(t) = \sum_\ell {\alpha_\ell}' t^{2\ell} + \sum_m {\beta_m}'(x+y)t^{2m+1}$. Then obviously $\varphi(t) + \varphi'(t)$ is split. Further,

$$\begin{aligned} \varphi(t)\varphi'(t) = &\sum_{j,\ell} \alpha_j {\alpha_\ell}' t^{2j+2\ell} + \sum_{j,m} \alpha_j {\beta_m}'(x+y)t^{2j+2m+1} \\ &+ \sum_{k,\ell} \beta_k {\alpha_\ell}'(x+y)t^{2k+2\ell+1} + \sum_{k,m} \beta_k {\beta_m}'(x+y)(x+y)t^{2k+2m+2}. \end{aligned}$$

Since $(x+y)(x+y) = 2 + xy + yx = (2+b_2)1$ by (3.4) and (3.1)(III)(9), it follows that $\varphi(t)\varphi'(t)$ is split. □

Next, to obtain the proposition corresponding to Lemma 4.5 in [7], we define the polynomials over $\widetilde{A}(\omega)$.

Let $Q_k(t) = 1 + (yx)t^2 + (yx)^2 t^4 + \cdots + (yx)^k t^{2k}$ and
$P_k(t) = 1 + (xy)t^2 + (xy)^2 t^4 + \cdots + (xy)^k t^{2k}$. Note $Q_k(t) = yP_k(t)y$. The following proposition is a slight modification of Lemma 4.5 in [7].

Proposition 4.3. *Let $p = 2n+1$.*
(1) $(y^{-1}t^{-1})(1-yt)Q_{2n}(t)yt(1-xt) \in S(t)$.

(2) $(y^{-1}t^{-1})\{(1-yt)Q_n(t)yt+(yx)^{n+1}t^{2n+2}\}(1-xt) \in S(t)$.
(3) $(y^{-1}t^{-1})\{(1-yt)Q_{3n+1}(t)yt+(yx)^{3n+2}t^{6n+4}\}(1-xt) \in S(t)$.
(4) $(y^{-1}t^{-1})(1-yt)Q_{4n}(t)yt(1-xt) \in S(t)$.

Proof. First we prove (2). Since

$$\begin{aligned}(1-yt)Q_n(t)yt+(yx)^{n+1}t^{2n+2} &= (1-yt)yP_n(t)t+(yx)^{n+1}t^{2n+2}\\ &= yt(1-yt)P_n(t)+yt(xy)^n xt^{2n+1}\\ &= yt\{(1-yt)P_n(t)+(xy)^n xt^{2n+1}\},\end{aligned}$$

it suffices to show

$$\{(1-yt)P_n(t)+(xy)^n xt^{2n+1}\}(1-xt) \in S(t). \tag{4.1}$$

Now a simple computation shows that

$$\begin{aligned}&\{(1-yt)P_n(t)+(xy)^n xt^{2n+1}\}(1-xt)\\ &= \left\{\sum_{k=0}^{n}(xy)^k t^{2k} - \sum_{k=0}^{n-1} y(xy)^k t^{2k+1}\right\}(1-xt)\\ &= 1+\sum_{k=1}^{n}\left\{(xy)^k+(yx)^k\right\}t^{2k} - \sum_{k=0}^{n-1}\left\{y(xy)^k+(xy)^k x\right\}t^{2k+1}\\ &= 1+\sum_{k=1}^{n}(a_{k-1}+a_k)t^{2k} - \sum_{k=0}^{n-1}(x+y)a_k t^{2k+1} \in S(t),\end{aligned}$$

by (3.4) and (3.5). This proves (4.1).

Proof of (1). Since

$$\begin{aligned}(1-yt)Q_{2n}(t)yt(1-xt) &= (1-yt)yP_{2n}(t)t(1-xt)\\ &= yt(1-yt)P_{2n}(t)(1-xt),\end{aligned}$$

it suffices to show

$$(1-yt)P_{2n}(t)(1-xt) \in S(t). \tag{4.2}$$

However, the following straightforward calculation proves (4.2):

$$\begin{aligned}&(1-yt)P_{2n}(t)(1-xt)\\ &= \sum_{k=0}^{2n}(xy)^k t^{2k} - \sum_{k=0}^{2n} y(xy)^k t^{2k+1} - \sum_{k=0}^{2n}(xy)^k xt^{2k+1} + \sum_{k=0}^{2n}(yx)^{k+1}t^{2k+2}\\ &= 1+\sum_{k=1}^{2n}\left\{(xy)^k+(yx)^k\right\}t^{2k} + (yx)^p t^{2p} - \sum_{k=0}^{2n}\left\{y(xy)^k+(xy)^k x\right\}t^{2k+1}\\ &= 1+\sum_{k=1}^{2n}(a_{k-1}+a_k)t^{2k} + t^{2p} - \sum_{k=0}^{2n} a_k(x+y)t^{2k+1} \in S(t).\end{aligned}$$

Proof of (3). Since

$$\begin{aligned}&\left\{(1-yt)Q_{3n+1}(t)yt+(yx)^{3n+2}t^{6n+4}\right\}(1-xt)\\ &= yt\{(1-yt)P_{3n+1}(t)+(xy)^{3n+1}xt^{6n+3}\}(1-xt),\end{aligned}$$

it suffices to show

$$\{(1-yt)P_{3n+1}(t)+(xy)^{3n+1}xt^{6n+3}\}(1-xt)\in S(t). \tag{4.3}$$

Since $P_{3n+1}(t)=P_{2n}(t)+t^{4n+2}P_n(t)$ and $(xy)^{3n+1}x=(xy)^n x$, we must show $\Big\{(1-yt)\{P_{2n}(t)+P_n(t)t^{4n+2}\}+(xy)^n xt^{6n+3}\Big\}(1-xt)\in S(t)$. However, since $(1-yt)P_{2n}(t)(1-xt)\in S(t)$ by (4.2), it suffices to show that

$$\{(1-yt)P_n(t)t^{4n+2}+(xy)^n xt^{6n+3}\}(1-xt)\in S(t). \tag{4.4}$$

Now, (4.4) follows from (4.1), since t^{4n+2} is split.

Proof of (4). Since $(yx)^{2n+1}=1$, we have

$$Q_{4n}(t)=\sum_{k=0}^{2n}(yx)^k t^{2k}+\sum_{k=2n+1}^{4n}(yx)^k t^{2k}=(1+t^{2p})Q_{2n}(t).$$

Since $(1+t^{2p})$ is split, it follows that

$$(y^{-1}t^{-1})(1-yt)Q_{4n}(t)yt(1-xt)=(1+t^{2p})(y^{-1}t^{-1})(1-yt)Q_{2n}(t)yt(1-xt)$$

is split by (1). □

5. Proof of Theorem 2.2.(I)

In this section we prove Theorem 2.2 (2.11) for a torus knot $K(1/p)$, $p=2n+1$ a prime. First we define various homomorphisms among group rings.

Let $g=x^{m_1}y^{m_2}x^{m_3}y^{m_4}\cdots x^{m_{k-1}}y^{m_k}$, where m_j are integers and let $m=\sum_{j=1}^{k}m_j$ and ℓ is arbitrary. Then we have:

(1) $\rho^*:\mathbb{Z}G(K)\to\mathbb{Z}D_p[t^{\pm1}]$ is defined by $\rho^*(g)=\rho(g)t^m$,

(2) $\zeta^*:\mathbb{Z}D_p[t^{\pm1}]\to\widetilde{A}(\omega)[t^{\pm1}]$ is defined by $\zeta^*(gt^\ell)=\zeta(g)t^\ell$,

(3) $\xi^*:\widetilde{A}(\omega)[t^{\pm1}]\to M_{2,2}(\mathbb{Z}[\omega][t^{\pm1}])$ is defined by $\xi^*(gt^\ell)=\xi(g)t^\ell$,

(4) $\gamma^*:M_{2,2}(\mathbb{Z}[\omega][t^{\pm1}])\to M_{2n,2n}(\mathbb{Z}[t^{\pm1}])$ is defined by

$$\gamma^*\begin{bmatrix}\sum_j p_jt^j & \sum_j q_jt^j\\ \sum_j r_jt^j & \sum_j s_jt^j\end{bmatrix}=\begin{bmatrix}\sum_j \gamma(p_j)t^j & \sum_j \gamma(q_j)t^j\\ \sum_j \gamma(r_j)t^j & \sum_j \gamma(s_j)t^j\end{bmatrix}. \tag{5.1}$$

Now we show the following proposition.

Proposition 5.1. *Let $p=2n+1$, a prime. Then $D_{\tau,K(1/p)}(t)$ is of the form $q(t)q(-t)$ for some integer polynomial $q(t)$.*

Proof. We write $G(K(1/p))=\langle x,y|R_0=W_0xW_0{}^{-1}y^{-1}=1\rangle$, where $W_0=(xy)^n$. Consider the free derivative of R_0 with respect to x;

$$\frac{\partial R_0}{\partial x}=(1-y)\frac{\partial W_0}{\partial x}+W_0=(1-y)\sum_{k=0}^{n-1}(xy)^k+(xy)^n,$$

and we write

$$\Phi^*\left(\frac{\partial R_0}{\partial x}\right) = \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix},$$

where $\Phi^* = \rho^* \circ \zeta^* \circ \xi^*$.
Then we see;

$$\begin{aligned}
&(1)\ h_{11}(t) = \sum_{k=0}^{n} a_k t^{2k} + \sum_{k=0}^{n-1} a_k t^{2k+1} = (1+t)\sum_{k=0}^{n-1} a_k t^{2k} + a_n t^{2n},\\
&(2)\ h_{12}(t) = \sum_{k=0}^{n} b_k t^{2k} + \sum_{k=0}^{n-1} b_k t^{2k+1} = (1+t)\sum_{k=0}^{n-1} b_k t^{2k} + b_n t^{2n},\\
&(3)\ h_{21}(t) = \sum_{k=0}^{n} c_k t^{2k} - \omega\sum_{k=0}^{n-1} a_k t^{2k+1} - \sum_{k=0}^{n-1} c_k t^{2k+1}\\
&\qquad\quad = -\omega t\sum_{k=0}^{n-1} a_k t^{2k} + (1-t)\sum_{k=0}^{n-1} c_k t^{2k} + c_n t^{2n},\\
&(4)\ h_{22}(t) = \sum_{k=0}^{n} d_k t^{2k} - \omega\sum_{k=0}^{n-1} b_k t^{2k+1} - \sum_{k=0}^{n-1} d_k t^{2k+1}\\
&\qquad\quad = -\omega t\sum_{k=0}^{n-1} b_k t^{2k} + (1-t)\sum_{k=0}^{n-1} d_k t^{2k} + d_n t^{2n}.
\end{aligned} \tag{5.2}$$

Since $h_{11}(1) = 0$ and $h_{21}(1) = 0$, both $h_{11}(t)$ and $h_{21}(t)$ are divisible by $1-t$. In fact, we have:

$$\begin{aligned}
h_{11}(t) &= (1-t)\Big\{\ \sum_{k=0}^{n-1}(2a_0 + 2a_1 + \cdots + 2a_{k-1} + a_k)t^{2k}\\
&\qquad + \sum_{k=0}^{n}(2a_0 + 2a_1 + \cdots + 2a_k)t^{2k+1}\Big\}\\
&= (1-t)\left\{\sum_{k=0}^{n-1}(b_k + b_{k+1})t^{2k} + \sum_{k=0}^{n-1} 2b_{k+1}t^{2k+1}\right\},\ \text{and}\\
h_{21}(t) &= -\omega t(1-t^2)\sum_{k=0}^{n-2}(a_0 + a_1 + \cdots + a_k)t^{2k}\\
&\qquad - \omega t(1-t)(a_0 + a_1 + \cdots + a_{n-1})t^{2n-2} + (1-t)\sum_{k=1}^{n-1} c_k t^{2k}\\
&= (1-t)\left\{-\omega t(1+t)\sum_{k=0}^{n-2} b_{k+1}t^{2k} - \omega t b_n t^{2n-2} + \sum_{k=1}^{n-1} c_k t^{2k}\right\}.
\end{aligned}$$

Since $c_k = \omega b_k$, we see $h_{21}(t) = (1-t)\left\{-\omega t\sum_{k=0}^{n-1} b_{k+1}t^{2k}\right\}$, and hence,

$$\frac{1}{1-t}\det\left(\frac{\partial R_0}{\partial x}\right)^{\Phi^*} = \frac{1}{1-t}\det\begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix} = \det\begin{bmatrix} h'_{11}(t) & h_{12}(t) \\ h'_{21}(t) & h_{22}(t) \end{bmatrix},$$

where

$$h_{11}'(t) = \sum_{k=0}^{n-1}(b_k + b_{k+1})t^{2k} + \sum_{k=0}^{n-1} 2b_{k+1}t^{2k+1}$$
$$= \sum_{k=1}^{n-1} b_k(1+t^2)t^{2k-2} + b_n t^{2n-2} + \sum_{k=1}^{n} 2b_k t^{2k-1}, \text{ and}$$
$$h_{21}'(t) = -\omega t \sum_{k=0}^{n-1} b_{k+1}t^{2k}.$$

Let $g(t) = \sum_{k=1}^{n-1} b_k(1+t^2)t^{2k+2} + b_n t^{2n-2}$ and $h(t) = \sum_{k=1}^{n} b_k t^{2k-1}$. Then

$$h_{11}'(t) = g(t) + 2h(t) \text{ and } h_{21}'(t) = -\omega h(t).$$

Further a straightforward computation shows that

$$h_{11}'(t) + h_{12}(t) = (1+t)(g(t)+h(t)).$$

And,

$$h_{21}'(t) + h_{22}(t) = -\omega t \sum_{k=1}^{n} b_k t^{2k-2} - \omega t \sum_{k=1}^{n-1} b_k t^{2k} + (1-t)\sum_{k=1}^{n-1} d_k t^{2k} + d_n t^{2n}$$
$$= -\sum_{k=1}^{n} c_k t^{2k-1} - \sum_{k=1}^{n-1} c_k t^{2k+1} + (1-t)\sum_{k=0}^{n-1} d_k t^{2k} + d_n t^{2n}.$$

Since $c_k + d_k = a_k$ and $d_0 = a_0$, we see

$$-\sum_{k=1}^{n-1} c_k t^{2k+1} - \sum_{k=0}^{n-1} d_k t^{2k+1} = -\sum_{k=0}^{n-1} a_k t^{2k+1},$$

and hence

$$h_{21}'(t) + h_{22}(t) = \sum_{k=0}^{n} d_k t^{2k} - \sum_{k=0}^{n-1}(a_k + c_{k+1})t^{2k+1}.$$

Now, $h_{21}'(t) + h_{22}(t)$ is divisible by $1+t$, and in fact, we have

$$h_{21}'(t) + h_{22}(t) = (1+t)\{g(t) - 2h(t) - \omega h(t)\}.$$

Therefore,

$$\frac{1}{(1-t)(1+t)} \det\left(\Phi^* \frac{\partial R_0}{\partial x}\right) = \det\begin{bmatrix} g(t)+2h(t) & g(t)+h(t) \\ -\omega h(t) & g(t)-2h(t)-\omega h(t) \end{bmatrix}$$
$$= \det\begin{bmatrix} g(t)+2h(t) & -h(t) \\ -\omega h(t) & g(t)-2h(t) \end{bmatrix},$$

and hence

$$\widetilde{\Delta}_{\tau,K(1/p)}(t) = g(t)^2 - (4+w)h(t)^2. \tag{5.3}$$

Now we apply the following key lemma.

Lemma 5.2. *Let C_n be the companion matrix of $\theta_n(z)$, the minimal polynomial of ω. Then there exists a matrix $V_n \in GL(n,\mathbb{Z})$ such that ${V_n}^2 = 4E_n + C_n$.*

Since our proof involves a lot of computations, the proof is postponed to Section 11.

Since the total twisted Alexander polynomial of $K(1/p)$ at τ is $D_{\tau,K(1/p)}(t) = \det[\widetilde{\Delta}_{\tau,K(1/p)}(t)]^{\gamma^*}$, we obtain, noting that V_n commutes with C_n,

$$\begin{aligned} D_{\tau,K(1/p)}(t) &= \det[g(t|C_n)^2 - {V_n}^2 h(t|C_n)^2] \\ &= \det[g(t|C_n) - V_n h(t|C_n)] \det[g(t|C_n) + V_n h(t|C_n)]. \end{aligned}$$

Let $q(t) = \det[g(t|C_n) - V_n h(t|C_n)]$. Then since $g(-t) = g(t)$ and $h(-t) = -h(t)$, it follows that

$$D_{\tau,K(1/p)}(t) = q(t)q(-t).$$

This proves Theorem 2.2(2.11) for $K(1/p)$.

Remark 5.3. It is quite likely that

$$q(t) = (1+t)^n \left\{\Delta_{K(1/p)}(t)\right\}^{n-1}, \tag{5.4}$$

where $\Delta_{K(1/p)}(t)$ is the Alexander polynomial of $K(1/p)$.

6. Proof of Theorem 2.2 (II)

Now we return to a proof of Theorem 2.2 (2.11) for a 2-bridge knot $K(r)$ in $H(p)$. Let $G(K(r)) = \langle x, y | R \rangle, R = WxW^{-1}y^{-1}$, be a Wirtinger presentation of $G(K(r))$. Then as is shown in [7], R is written freely as a product of conjugates of R_0: $R = \prod_{j=1}^{s} u_j {R_0}^{\epsilon_j} {u_j}^{-1}$, where for $1 \le j \le s$, $\epsilon_j = \pm 1$ and $u_j \in F(x,y)$, the free group generated by x and y, and $\frac{\partial R}{\partial x} = \sum_j \epsilon_j u_j (\frac{\partial R_0}{\partial x})$, and hence

$$\begin{aligned} \widetilde{\Delta}_{\tau,K(r)}(t) &= \det\left(\frac{\partial R}{\partial x}\right)^{\Phi^*} / \det(y^{\Phi^*} - E_2) \\ &= \widetilde{\Delta}_{\tau,K(1/p)}(t) \det\Big(\sum_j \epsilon_j u_j\Big)^{\Phi^*}. \end{aligned}$$

As we did in [7], we study $\lambda(r) = (\sum_j \epsilon_j u_j)^{\tau^*} \in \widetilde{A}(\omega)[t^{\pm 1}]$, where $\tau^* = \rho^* \circ \zeta^*$. For simplicity, we denote $\tau^*(\lambda(r))$ by ${\lambda_r}^*(t)$. In fact, it is a polynomial in $t^{\pm 1}$. Since $K(r) \in H(p)$, the continued fraction of r is of the form: $r = [pk_1, 2m_1, pk_2, \cdots, 2m_\ell, pk_{\ell+1}]$, where k_j and m_j are non-zero integers. First we state the following proposition.

Proposition 6.1. *Suppose $K(r)$ and $K(r')$ belong to $H(p)$ and let $r = [pk_1, 2m_1, pk_2, \cdots, 2m_\ell, pk_{\ell+1}]$, $r' = [p{k_1}', 2m_1, p{k_2}', \cdots, 2m_\ell, pk'_{\ell+1}]$ be continued fractions of r and r'. Suppose that $k_j \equiv {k_j}'$ (mod 4) for each $j, 1 \le j \le \ell+1$. Then if $y^{-1}t^{-1}\lambda^*_r(t)$ is split, so is $y^{-1}t^{-1}\lambda^*_{r'}(t)$.*

Since a proof is analogous to that of Proposition 6.3 in [7], we omit the details.

Now we study the polynomial $\lambda_r{}^*(t) \in \widetilde{A}(\omega)[t^{\pm 1}]$ and we prove that $y^{-1}t^{-1}\lambda_r{}^*(t)$ is split. As is seen in Section 7 in [7], $\lambda_r{}^*(t)$ is written as $w^*{}_{2\ell+1}(t)$ and we will prove the following proposition. The same notation employed in Section 7 in [7] will be used in this section.

Proposition 6.2. $y^{-1}t^{-1}w^*_{2\ell+1}(t) \in S(t)$.

Proof. Use induction on j. First we prove $y^{-1}t^{-1}w^*{}_1(t) \in S(t)$.
(1) If $w_1(t) = yt$, then $y^{-1}t^{-1}w^*{}_1(t) = 1$ and hence $y^{-1}t^{-1}w^*{}_1(t) \in S(t)$.
(2) If $w_1 = y - (yx)^{n+1}$, then $w^*{}_1(t) = yt - (yx)^{n+1}t^{2n+2}$ and
$y^{-1}t^{-1}w^*{}_1(t) = 1 - (xy)^n xt^{2n+1} = 1 + b_n(x+y)t^{2n+1}$ and hence
$y^{-1}t^{-1}w^*{}_1(t) \in S(t)$.
(3) If $w_1 = -(yx)^{n+1}$, then $w^*{}_1(t) = -(yx)^{n+1}t^{2n+2}$ and
$y^{-1}t^{-1}w^*{}_1(t) = -(xy)^n xt^{2n+1} = b_n(x+y)t^{2n+1}$ and hence
$y^{-1}t^{-1}w^*{}_1(t) \in S(t)$.

Now suppose $y^{-1}t^{-1}w^*{}_{2j-1}(t) \in S(t)$ for $j \leq \ell$, and we claim $y^{-1}t^{-1}w^*{}_{2\ell+1}(t) \in S(t)$. There are three cases to be considered. (See [7, Proposition 7.1.]

Case 1. $k_{\ell+1} = 1$. $w_{2\ell+1} = \{(1-y)Q_n y + (yx)^{n+1}\}\sum_j m_j(x-1)y^{-1}w_{2j-1} - (yx)^{n+1}y^{-1}w_{2\ell-1} + y$.
Then

$$\begin{aligned} y^{-1}t^{-1}w^*{}_{2\ell+1}(t) = y^{-1}t^{-1}\{&(1-yt)Q_n(t)yt \\ &+ (yx)^{n+1}t^{2n+2}\}\sum_j m_j(xt-1)y^{-1}t^{-1}w^*{}_{2j-1}(t) \\ &- (xy)^n xt^{2n+1}(y^{-1}t^{-1}w^*{}_{2\ell-1}(t)) + 1. \end{aligned}$$

By Proposition 4.3(2), each summand is split. Further, $-(xy)^n xt^{2n+1} = b_n(x+y)t^{2n+1} \in S(t)$ and $1 \in S(t)$. Therefore, the sum of them is split.

Proofs of the other cases are essentially the same.

Case 2. $k_{\ell+1} = 2$.
$w_{2\ell+1} = (1-y)Q_{2n}y\{\sum_j m_j(x-1)y^{-1}w_{2j-1}\} + (yx)^{2n+1}w_{2\ell-1} - (yx)^{n+1} + y$.
Then

$$\begin{aligned} y^{-1}t^{-1}w^*{}_{2\ell+1}(t) = y^{-1}t^{-1}&(1-yt)Q_{2n}(t)yt\{\sum_j m_j(xt-1)y^{-1}t^{-1}w^*{}_{2j-1}(t)\} \\ &+ y^{-1}t^{-1}t^{4n+2}w^*{}_{2\ell-1}(t) - x(yx)^n t^{2n+1} + 1. \end{aligned}$$

Again, $y^{-1}t^{-1}(1-yt)Q_{2n}(t)yt(xt-1) \in S(t)$ by Proposition 4.3(1) and $y^{-1}t^{-1}w^*{}_{2j-1}(t) \in S(t)$ by induction hypothesis and t^{4n+2}, $-x(yx)^n t^{2n+1} = b_n(x+y)t^{2n+1}$ and 1 are split. Thus, $y^{-1}t^{-1}w^*{}_{2\ell+1}(t) \in S(t)$.

Case 3. $k_{\ell+1} = 3$.

$$\begin{aligned} w_{2\ell+1} = \{(1-y)Q_{3n+1}y + (yx)^{3n+2}\} \sum_j m_j(x-1)y^{-1}w_{2j-1} \\ - (yx)^{3n+2}y^{-1}w_{2\ell-1} + (yx)^p y - (yx)^{n+1} + y. \end{aligned}$$

Then

$$\begin{aligned} y^{-1}t^{-1}w^*{}_{2\ell+1}(t) = y^{-1}t^{-1}\{&(1-yt)Q_{3n+1}(t)yt \\ &+ (yx)^{3n+2}t^{6n+4}\} \sum_j m_j(xt-1)y^{-1}t^{-1}w^*{}_{2j-1}(t) \\ &- (xy)^n xt^{6n+3}(y^{-1}t^{-1}w^*{}_{2\ell-1}(t)) + t^{2p} - (xy)^n xt^{2n+1} + 1. \end{aligned}$$

We see that $y^{-1}t^{-1}w^*{}_{2\ell+1}(t)$ is split, since each of $y^{-1}t^{-1}\{(1-yt)Q_{3n+1}(t)yt + (yx)^{3n+2}t^{6n+4}\}(xt-1)$, $y^{-1}t^{-1}w^*{}_{2j-1}(t)$ and $-(xy)^n xt^{6n+3} = b_n(x+y)t^{6n+3}$ and $-(xy)^n xt^{2n+1} = b_n(x+y)t^{2n+1}$ is split. This proves Proposition 6.1 □

Now a proof of (2.11) for our knots is exactly the same as we did in Section 5. Since $y^{-1}t^{-1}w^*{}_{2\ell+1}(t) \in S(t)$, we can write

$$y^{-1}t^{-1}w^*{}_{2\ell+1}(t) = \sum_j \alpha_j t^{2j} + \sum_k \beta_k(x+y)t^{2k+1},$$

where $\alpha_j, \beta_k \in \mathbb{Z}[\omega]$.

Define $g(t) = \sum_j \alpha_j t^{2j}$ and $h(t) = \sum_k \beta_k t^{2k+1}$. Since $X + Y = \begin{bmatrix} -2 & 1 \\ \omega & 2 \end{bmatrix}$,

$$\xi^*[y^{-1}t^{-1}w^*{}_{2\ell+1}(t)] = \begin{bmatrix} g(t) - 2h(t) & h(t) \\ \omega h(t) & g(t) + 2h(t) \end{bmatrix} \text{ and}$$

$$\det(y^{-1}t^{-1}w^*{}_{2\ell+1}(t))^{\xi^*} = g(t)^2 - (\omega+4)h(t)^2.$$

Thus $\widetilde{\Delta}_{\tau,K(r)}(t|\omega) = \widetilde{\Delta}_{\tau,K(1/p)}(t|\omega)\{g(t)^2 - (w+4)h(t)^2\}$, and hence, we have

$$D_{\tau,K(r)}(t) = D_{\tau,K(1/p)}(t)\det[g(t|C_n)^2 - (C_n + 4E_n)h(t|C_n)^2].$$

Now by Lemma 5.2, there exists a matrix $V_n \in GL(n,\mathbb{Z})$ such that ${V_n}^2 = C_n + 4E_n$. Since V_n commutes with C_n, we see

$$\begin{aligned} g(t|C_n)^2 - (C_n + 4E_n)h(t|C_n)^2 &= g(t|C_n)^2 - {V_n}^2 h(t|C_n)^2 \\ &= \{g(t|C_n) - V_n h(t|C_n)\}\{g(t|C_n) + V_n h(t|C_n)\}. \end{aligned}$$

Let $f(t) = \det[g(t|C_n) - V_n h(t|C_n)]$. Since $h(-t|C_n) = -h(t|C_n)$ and $g(-t|C_n) = g(t|C_n)$, $f(-t) = \det[g(t|C_n) + V_n h(t|C_n)]$, and thus,

$$\det[g(t|C_n)^2 - (C_n + 4E_n)h(t|C_n)^2] = f(t)f(-t).$$

Therefore, $D_{\tau,K(r)}(t) = D_{\tau,K(1/p)}(t)f(t)f(-t)$. Since $D_{\tau,K(1/p)}(t)$ is of the form $q(t)q(-t)$, it follows that $D_{\tau,K(r)}(t) = F(t)F(-t)$, where $F(t) = q(t)f(t)$.

This proves (2.11) for $K(r)$ in $H(p)$. □

7. Proof of Theorem 2.2 (III)

In this section, we prove (2.12) for a 2-bridge knot $K(r)$ with $\Delta_{K(r)}(-1) \equiv 0 \pmod{p}$.

First we state the following easy lemma without proof.

Lemma 7.1. *Let M be a $2n \times 2n$ matrix over a commutative ring which is decomposed into four $n \times n$ matrices, A, B, C and D: $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.*

Suppose that each matrix is lower triangular and in particular, C is strictly lower triangular, namely, all diagonal entries are 0. Then $\det M = (\det A)(\det D)$, and hence, $\det M$ is the product of all diagonal entries of M.

Lemma 7.1 can be proven easily by induction on n.

Now let $K(r), 0 < r < 1$, be a 2-bridge knot and consider a Wirtinger presentation $G(K(r)) = \langle x, y | R\rangle$, where $R = x^{\epsilon_1}y^{\eta_1}x^{\epsilon_2}y^{\eta_2}\cdots x^{\epsilon_\alpha}y^{\eta_\alpha}$ and $\epsilon_j, \eta_j = \pm 1$ for $1 \le j \le \alpha$.

Applying the free differentiation, we have $\dfrac{\partial R}{\partial x} = \sum\limits_{i=1}^{\alpha} g_i, g_i \in \mathbb{Z}G(K)$, where

$$g_i = \begin{cases} x^{\epsilon_1}y^{\eta_1}x^{\epsilon_2}y^{\eta_2}\cdots x^{\epsilon_{i-1}}y^{\eta_{i-1}} & \text{if } \epsilon_i = 1 \\ -x^{\epsilon_1}y^{\eta_1}x^{\epsilon_2}y^{\eta_2}\cdots x^{\epsilon_{i-1}}y^{\eta_{i-1}}x^{-1} & \text{if } \epsilon_i = -1. \end{cases} \tag{7.1}$$

Let $\Psi : \mathbb{Z}G(K) \to \mathbb{Z}[t^{\pm 1}]$ be the homomorphism defined by $\Psi(g_i) = \epsilon_i t^{m_i}$, where $m_i = \sum\limits_{j=1}^{i-1}(\epsilon_j + \eta_j) + \dfrac{\epsilon_i - 1}{2}$.

Then $\left(\dfrac{\partial R}{\partial x}\right)^{\Psi}$ gives the Alexander polynomial $\Delta_{K(r)}(t)$ of $K(r)$. On the other hand, $\dfrac{1}{(1-t)(1+t)}\det\left(\dfrac{\partial R}{\partial x}\right)^{\Phi^*}$ gives the twisted Alexander polynomial $\widetilde{\Delta}_{\rho_0,K(r)}(t|w)$ associated to the irreducible dihedral representation ρ_0, and further, we see $D_{\tau,K(r)}(t) = \det\left[\dfrac{1}{(1-t)(1+t)}(\dfrac{\partial R}{\partial x})^{\Phi^*}\right]^{\gamma^*}$.

Now using (7.1), we compute $\left(\dfrac{\partial R}{\partial x}\right)^{\Phi^*} = \sum_i \Phi^*(g_i)$.

If $\epsilon_i = 1$, then m_i is even and

$$\begin{aligned} \Phi^*(g_i) &= [(xy)^{i-1}]^{\xi} t^{m_i} \\ &= \begin{bmatrix} a_{i-1} & b_{i-1} \\ c_{i-1} & d_{i-1} \end{bmatrix} t^{m_i}. \end{aligned}$$

If $\epsilon_i = -1$, then m_i is odd and

$$\begin{aligned} \Phi^*(g_i) &= -[(xy)^{i-1}x]^{\xi} t^{m_i} \\ &= -\begin{bmatrix} -a_{i-1} & a_{i-1} + b_{i-1} \\ -c_{i-1} & c_{i-1} + d_{i-1} \end{bmatrix} t^{m_i}. \end{aligned}$$

Therefore we have

$$\begin{aligned}(\frac{\partial R}{\partial x})^{\Phi^*} &= \sum_i \Phi^*(g_i)\\ &= \sum_{m_i=even} \begin{bmatrix} a_{i-1} & b_{i-1} \\ c_{i-1} & d_{i-1} \end{bmatrix} t^{m_i} - \sum_{m_j=odd} \begin{bmatrix} -a_{j-1} & a_{j-1}+b_{j-1} \\ -c_{j-1} & c_{j-1}+d_{j-1} \end{bmatrix} t^{m_j}.\end{aligned}$$

We note that as polynomials on ω, the constant terms of a_{i-1} and d_{i-1} both are 1. Further, since $c_{i-1} = \omega b_{i-1}$, the constant term of $c_{i-1}+d_{i-1}$ is also 1, and hence

$$\sum_i [{g_i}^{\Phi^*}]^{\gamma^*} = \begin{bmatrix} \Delta_{K(r)}(-t)+\omega\mu_{11} & \mu_{12} \\ \omega\mu_{21} & \Delta_{K(r)}(t)+\omega\mu_{11} \end{bmatrix}, \text{ where } \mu_{ij} \in (\mathbb{Z}[\omega])[t^{\pm 1}].$$

If we replace $\mathbb{Z}$ by $\mathbb{Z}/p$, then C_n is reduced to $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \hline & & & 0 \\ & E & & \vdots \\ & & & 0 \end{array}\right]$ and hence $\sum_i [{g_i}^{\Phi^*}]^{\gamma^*} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (mod p), where A, B, C and D are lower triangular and in particular, C is strictly lower triangular, and each diagonal entry of A and D is $\Delta_{K(r)}(t)$ (mod p) and $\Delta_{K(r)}(-t)$ (mod p), respectively. Therefore, by Lemma 7.1, we have

$$\begin{aligned} D_{\tau,K(r)}(t) &\equiv \det(\sum_i [{g_i}^{\Phi^*}]^{\gamma^*}) / \det[(1-t)(1+t)]^{\gamma^*} \\ &\equiv \left\{\frac{\Delta_{K(r)}(t)}{1+t}\right\}^n \left\{\frac{\Delta_{K(r)}(-t)}{1-t}\right\}^n \pmod{p}. \end{aligned}$$

This proves (2.12) for any 2-bridge knot $K(r)$ with $\alpha \equiv 0$ (mod p). We note that $\Delta_{K(r)}(t)$ is divisible by $1+t$ over $(\mathbb{Z}/p)[t^{\pm 1}]$.

8. $N(q,p)$-representations

In this section, we discuss another type of metacyclic representations and the twisted Alexander polynomial associated to these representations. Let $q \geq 1$ and $p = 2n+1$ be an odd prime. Consider a metacyclic group, $N(q,p) = \mathbb{Z}/2q \ltimes \mathbb{Z}/p$ that is a semi-direct product of $\mathbb{Z}/2q$ and $\mathbb{Z}/p$ defined by

$$N(q,p) = \langle s, a | s^{2q} = a^p = 1, sas^{-1} = a^{-1}\rangle. \tag{8.1}$$

Note that $N(1,p) = D_p$ and $N(2,p)$ is a binary dihedral group, denoted by N_p. Since s^2 generates the center of $N(q,p)$, we see that $N(q,p)/\langle s^2\rangle = D_p$ and hence $|N(q,p)| = 2pq$. For simplicity, we assume hereafter that $\gcd(q,p) = 1$. Now it is known [6], [5] that the knot group $G(K)$ of a knot K is mapped onto $N(q,p)$ if and only if $G(K)$ is mapped onto D_p, namely, $\Delta_K(-1) \equiv 0$ (mod p). For a 2-bridge knot

$K(r)$, if $\Delta_{K(r)}(-1) \equiv 0 \pmod{p}$, then we may assume without loss of generality that there is an epimorphism $\widetilde{\rho} : G(K(r)) \longrightarrow N(q,p)$ for any $q \geq 1$ such that

$$\widetilde{\rho}(x) = s \text{ and } \widetilde{\rho}(y) = sa. \tag{8.2}$$

As before, we draw a diagram below consisting of various groups and connecting homomorphisms.

$$\begin{array}{ccccccc}
 & & & & GL(2qp,\mathbb{Z}) & & \\
 & & & \nearrow \widetilde{\xi} & & & \\
 & & N(q,p) & \xrightarrow{\widetilde{\pi}} & GL(2n,\mathbb{C}) & \xrightarrow{\widetilde{\gamma}} & GL(2nm,\mathbb{Z}) \\
 & \widetilde{\rho} \nearrow & & & & & \\
G(K) & \xrightarrow{\rho_p} & N_p & \xrightarrow{\xi_p} & SU(2,\mathbb{C}) & \xrightarrow{\gamma_p} & GL(4n,\mathbb{Z}) \\
 & \rho \searrow & & & & & \\
 & & D_p & \xrightarrow{\pi_0} & GL(2n,\mathbb{Z}) & & \\
 & & & \searrow \pi & & & \\
 & & & & GL(p,\mathbb{Z}) & &
\end{array}$$

Here, $p = 2n+1, \widehat{\rho} = \rho \circ \pi, \rho_0 = \rho \circ \pi_0, \widetilde{\nu} = \widetilde{\rho} \circ \widetilde{\xi}, \widetilde{\tau} = \widetilde{\rho} \circ \widetilde{\pi}, \tau_p = \rho_p \circ \xi_p$ and m is the degree of the minimal polynomial of ζ over $\mathbb{Q}$.

Using the irreducible representation π_0 of D_p on $GL(2n,\mathbb{Z})$, we can define a representation of $N(q,p)$ on $GL(2n,\mathbb{C})$. In fact, we have

Lemma 8.1. *Let ζ be a primitive $2q$-th root of 1, $q \geq 1$. Then the mapping $\widetilde{\pi}$: $N(q,p) \longrightarrow GL(2n,\mathbb{C})$ defined by*

$$\begin{aligned} \widetilde{\pi}(s) &= \zeta \pi_0(x) \text{ and} \\ \widetilde{\pi}(sa) &= \zeta \pi_0(y) \end{aligned} \tag{8.3}$$

gives a representation of $N(q,p)$ on $GL(2n,\mathbb{C})$.

Since a proof is straightforward, we omit details. Now $\widetilde{\tau} = \widetilde{\rho} \circ \widetilde{\pi} : G(K(r)) \longrightarrow GL(2n,\mathbb{C})$ defines a metacyclic representation of $G(K(r))$. Then the twisted Alexander polynomial $\widetilde{\Delta}_{\widetilde{\tau},K(r)}(t|\zeta)$ of $K(r)$ associated to $\widetilde{\tau}$ is given by

$$\widetilde{\Delta}_{\widetilde{\tau},K(r)}(t|\zeta) = \widetilde{\Delta}_{\rho_0,K(r)}(\zeta t), \tag{8.4}$$

where $\rho_0 = \rho \circ \pi_0$.

Therefore, the total twisted Alexander polynomial is

$$D_{\widetilde{\tau},K(r)}(t) = \prod_{(2q,k)=1} \widetilde{\Delta}_{\rho_0,K(r)}(\zeta^k t). \tag{8.5}$$

This proves the following theorem.

Theorem 8.2. *Let $p = 2n+1$ be an odd prime and $q \geq 1$. Let $K(r)$ be a 2-bridge knot. Suppose $\Delta_{K(r)}(-1) \equiv 0$ (mod p). Then $G(K(r))$ has a metacyclic*

representation

$$\widetilde{\tau} = \widetilde{\rho} \circ \widetilde{\pi} : G(K(r)) \longrightarrow N(q,p) \longrightarrow GL(2n, \mathbb{C}).$$

Let ζ be a primitive $2q$-th root of 1. Then the twisted Alexander polynomial $\widetilde{\Delta}_{\widetilde{\tau},K(r)}(t)$ and the total twisted Alexander polynomial $D_{\widetilde{\tau},K(r)}(t)$ associated to $\widetilde{\tau}$ are given by

$$\begin{aligned}&(1)\ \widetilde{\Delta}_{\widetilde{\tau},K(r)}(t) = \widetilde{\Delta}_{\rho_0,K(r)}(\zeta t).\\ &(2)\ D_{\widetilde{\tau},K(r)}(t) = \prod_{(2q,k)=1} \widetilde{\Delta}_{\rho_0,K(r)}(\zeta^k t).\end{aligned} \tag{8.6}$$

We conclude this section with a few remarks. First, as we mentioned earlier, if $q = 2$, $N(2,p)$ is a binary dihedral group, denoted by N_p. It is known [9] [12] that generators s and sa of N_p are represented in $SU(2,\mathbb{C})$ by trace free matrices. In fact, the mapping ξ_p:

$$\xi_p(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \xi_p(sa) = \begin{bmatrix} 0 & v_p \\ -v_p^{-1} & 0 \end{bmatrix} \tag{8.7}$$

gives a representation of N_p into $SU(2,\mathbb{C})$, where $v_p = e^{\frac{2\pi i}{p}}$.

Then we will show that the total twisted Alexander polynomial $D_{\tau_p,K(r)}(t)$ associated to $\tau_p = \rho_p \circ \xi_p$ is given by

$$D_{\tau_p,K(r)}(t) = \widetilde{\Delta}_{\rho_0,K(r)}(it)\widetilde{\Delta}_{\rho_0,K(r)}(-it), \tag{8.8}$$

where $i = \sqrt{-1}$. Therefore we have the following corollary.

Corollary 8.3. *If $q = 2$, then $D_{\widetilde{\tau},K(r)}(t) = D_{\tau_p,K(r)}(t)$.*

Proof of (8.8). Let C_p be the companion matrix of the minimal polynomial of v_p, namely, $C_p = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & -1 \\ \hline & & & -1 \\ & E & & \vdots \\ & & & -1 \end{array}\right]$. Then, by definition, we have

$$D_{\tau_p,K(r)}(t) = \det[\widetilde{\Delta}_{\tau_p,K(r)}(t|C_p)]. \tag{8.9}$$

And (8.8) follows from the following lemma.

Lemma 8.4. *Let $E^*_{2n} = [a_{j,k}]$ be a $2n \times 2n$ matrix such that $a_{j,k} = 1$, if $k+j = 2n+1$ and 0, otherwise (E^*_{2n} is the 'mirror image' of E_{2n}.) Denote*

$$A = \begin{bmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & C_p \\ -C_p^{-1} & 0 \end{bmatrix}, \textit{ and}$$

$$\widehat{A} = \begin{bmatrix} iE^*_{2n} & 0 \\ 0 & -iE^*_{2n} \end{bmatrix}, \widehat{B} = \begin{bmatrix} i\pi_0(y) & 0 \\ 0 & -i\pi_0(y) \end{bmatrix}.$$

Then there exists a matrix $M_{4n} \in GL(4n, \mathbb{C})$ *such that* ${M_{4n} A M_{4n}}^{-1} = \widehat{A}$ *and* ${M_{4n} B M_{4n}}^{-1} = \widehat{B}$.

Proof. A simple computation shows that $M_{4n} = \frac{1}{\sqrt{2}} \begin{bmatrix} E_{2n} & -iE_{2n}^* \\ E_{2n} & iE_{2n}^* \end{bmatrix}$ is what we sought. □

Secondly, the metacyclic group $N(q,p)$ is also represented by $\widetilde{\xi}$ in $GL(2qp, \mathbb{Z})$ via *maximum* permutation representation on the symmetric group S_{2qp}. To be more precise, let

$$S = \{1, s, s^2, \cdots, s^{2q-1}, a, sa, s^2a, \cdots, s^{2q-1}a, a^2, sa^2, s^2a^2, \cdots, s^{2q-1}a^2, \cdots, \\ a^{p-1}, sa^{p-1}, s^2a^{p-1}, \cdots, s^{2q-1}a^{p-1}\}$$

be the ordered set of the elements of $N(q,p)$. Then the right multiplication by an element g of $N(q,p)$ on S induces a permutation associated to g, and by taking the permutation matrix corresponding to this permutation, we obtain the representation $\widetilde{\xi}$ of $N(q,p)$ on $GL(2qp, \mathbb{Z})$.

Then we have the following:

Proposition 8.5. *For any* $q \geq 1$, *the twisted Alexander polynomial* $\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t)$ *of* $K(r)$ *associated to* $\widetilde{\nu} = \widetilde{\rho} \circ \widetilde{\xi}$ *is given by*

$$\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t) = \frac{\displaystyle\prod_{k=0}^{2q-1} \Delta_{K(r)}(\zeta^k t)}{1 - t^{2q}} \prod_{k=0}^{2q-1} \widetilde{\Delta}_{\rho_0, K(r)}(\zeta^k t), \tag{8.10}$$

where ζ *is a primitive* $2q$*-th root of* 1. *Therefore,* $\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t)$ *is an integer polynomial in* t^{2q} *and* $D_{\widetilde{\tau}, K(r)}(t)$ *divides* $\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t)$.

Proof. By construction, $\widetilde{\xi}(s) = \rho(x) \otimes C$ and $\widetilde{\xi}(sa) = \rho(y) \otimes C$, where C is the transpose of the companion matrix of $t^{2q} - 1$ and $[a_{i,j}] \otimes C = [a_{i,j} C]$, the tensor product of $[a_{i,j}]$ and C.

Therefore (8.10) follows immediately. □

If Conjecture A holds for $K(r)$, $\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t)$ is of the form:

$$\widetilde{\Delta}_{\widetilde{\nu}, K(r)}(t) = \frac{\prod_{k=0}^{2q-1} \Delta_{K(r)}(\zeta^k t)}{1 - t^{2q}} f(t^{2q})^2,$$

for some integer polynomial $f(t^{2q})$ in t^{2q}.

If coefficients are taken from a finite field, then (8.10) becomes much simpler. The following proposition is a metacyclic version of (2.12). Since a proof is easy, we omit details.

Proposition 8.6. *Let* p *be an odd prime. Suppose* $\Delta_{K(r)}(-1) \equiv 0$ *(mod* p*). Then*

we have

$$\widetilde{\Delta}_{\widetilde{\nu},K(r)}(t) \equiv \left\{ \prod_{k=0}^{2q-1} \Delta_{K(r)}(\zeta^k t) \right\}^p / (1-t^{2q})^p \pmod{p}. \tag{8.11}$$

Remark 8.7. In [1], Cha defined the twisted Alexander invariant of a fibred knot K using its Seifert fibred surface. Evidently, this invariant is closely related to our twisted Alexander polynomial. For example, as is described in [1, Example], if we consider a regular dihedral representation ρ, then the invariant he defined is essentially the same as the twisted Alexander polynomial associated to a regular dihedral representation $\widetilde{\nu} = \widetilde{\rho} \circ \widetilde{\xi} : G(K) \to D_p = N(1,p) \to GL(2p, \mathbb{Z})$ we discussed in this section. More precisely, let $A_{\rho,K}(t)$ be Cha's twisted Alexander invariant associated to ρ and $\widetilde{\Delta}_{\widetilde{\nu},K}(t)$ the twisted Alexander polynomial of a knot K associated to $\widetilde{\nu}$. Then we have

$$(1-t^2)\widetilde{\Delta}_{\widetilde{\nu},K}(t) = A_{\rho,K}(t^2). \tag{8.12}$$

We should note that $\widetilde{\Delta}_{\widetilde{\nu},K}(t)$ is an integer polynomial in t^2. (See Proposition 8.5) Details will appear elsewhere.

9. Example

The following examples illustrate our main theorem.

Example 9.1. Dihedral representations $\tau : G(K(r)) \longrightarrow D_p \longrightarrow GL(2, \mathbb{C})$.

(I) Let $p = 3$ and $n = 1$. Then $\theta_1(z) = z + 3$ and $\omega = -3$.

(a) $r = 1/3$. $D_{\tau,K(1/3)}(t) = \widetilde{\Delta}_{\rho_0,K(1/3)}(t) = 1 - t^2$.

(b) $r = 1/9$. $D_{\tau,K(1/9)}(t) = \widetilde{\Delta}_{\rho_0,K(1/9)}(t) = (1-t^2)(1-t^3+t^6)(1+t^3+t^6)$.

(c) $r = 5/27$. $D_{\tau,K(5/27)}(t) = \widetilde{\Delta}_{\rho_0,K(r)}(t) = (1-t^2)(1+t-t^2+t^3+t^4)(1-t-t^2-t^3+t^4)$.

Note $\Delta_{K(5/27)}(t) = (1-t+t^2)(2-2t+t^2-2t^3+2t^4)$ and $2-2t+t^2-2t^3+2t^4 \equiv -(1-t-t^2-t^3+t^4) \pmod 3$, and

$$\begin{aligned} D_{\tau,K(5/27)}(t) &\equiv \frac{\Delta_{K(r)}(t)}{1+t}\frac{\Delta_{K(r)}(-t)}{1-t} \\ &\equiv \frac{(1+t)^2(1-t-t^2-t^3+t^4)}{1+t}\frac{(1-t)^2(1+t-t^2+t^3+t^4)}{1-t} \\ &\equiv (1-t^2)(1-t-t^2-t^3+t^4)(1+t-t^2+t^3+t^4) \pmod 3. \end{aligned}$$

(II) Let $p = 5$ and $n = 2$. Then $\theta_2(z) = z^2 + 5z + 5$.

(a) $r = 1/5$. $D_{\tau,K(r)}(t) = (1-t^2)^2\Delta_{K(1/5)}(t)\Delta_{K(1/5)}(-t)$.

(b) $r = 19/85$ $D_{\tau,K(r)}(t) = D_{\tau,K(1/5)}(t)f(t)f(-t)$, where $f(t) = 1-3t-2t^2+4t^3-t^4-4t^6-3t^7+7t^8-3t^9-4t^{10}-t^{12}+4t^{13}-2t^{14}-3t^{15}+t^{16}$, and $\Delta_{K(r)}(t) = \Delta_{K(1/5)}(t)g(t)$, where $g(t) = 2-2t+2t^2-2t^3+t^4-2t^5+2t^6-2t^7+2t^8$, and $f(t) \equiv g(t)^2 \pmod 5$.

Since $\Delta_{K(1/5)}(t) \equiv (1+t)^4 \pmod 5$, we see

$$\begin{aligned} D_{\tau,K(r)}(t) &= D_{\tau,K(1/5)}(t)f(t)f(-t) \\ &= \left\{(1+t)^2\Delta_{K(1/5)}(t)f(t)\right\}\left\{(1-t)^2\Delta_{K(1/5)}(-t)f(-t)\right\} \\ &\equiv \{(1+t)^6 g(t)^2\}\{(1-t)^6 g(-t)^2\} \\ &\equiv \{(1+t)^3 g(t)\}^2\{(1-t)^3 g(-t)\}^2 \\ &\equiv \left\{\frac{\Delta_{K(1/5)}(t)g(t)}{1+t}\right\}^2\left\{\frac{\Delta_{K(1/5)}(-t)g(-t)}{1-t}\right\}^2 \\ &\equiv \left\{\frac{\Delta_{K(r)}(t)}{1+t}\right\}^2\left\{\frac{\Delta_{K(r)}(-t)}{1-t}\right\}^2 \pmod 5. \end{aligned}$$

(c) $r = 21/115$. $D_{\tau,K(r)}(t) = D_{\tau,K(1/5)}(t)f(t)f(-t)$, where $f(t) = 4+2t-3t^2-t^3-8t^5-3t^6+4t^7+t^9+9t^{10}+t^{11}+4t^{13}-3t^{14}-8t^{15}-t^{17}-3t^{18}+2t^{19}+4t^{20}$, and $\Delta_{K(r)}(t) = \Delta_{K(1/5)}(t)g(t)$, where $g(t) = 2-2t+2t^2-2t^3+2t^4-3t^5+2t^6-2t^7+2t^8-2t^9+2t^{10}$, and $f(t) \equiv g(t)^2 \pmod 5$. Therefore, we see

$$D_{\tau,K(r)}(t) \equiv \left\{\frac{\Delta_{K(r)}(t)}{1+t}\right\}^2\left\{\frac{\Delta_{K(r)}(-t)}{1-t}\right\}^2 \pmod 5.$$

Example 9.2. Binary dihedral representations. $\tau_p : G(K(r)) \longrightarrow N_p \longrightarrow GL(2n,\mathbb{C})$

(I) Let $p=3$ and $n=1$.

(a)When $r = 1/9$, $D_{\tau_p,K(r)}(t) = (1+t^2)^2(1-t^6+t^{12})^2$.

(b)When $r = 5/27$, $D_{\tau_p,K(r)}(t) = (1+t^2)^2(1+3t^2+t^4+3t^6+t^8)^2$.

(II) Let $p=5$ and $n=2$.

(a) When $r = 1/5$, $D_{\tau_p,K(r)}(t) = (1+t^2)^4(1-t^2+t^4-t^6+t^8)^2$.

(b) When $r = 19/85$, $D_{\tau_p,K(r)}(t) = (1+t^2)^4(1-t^2+t^4-t^6+t^8)^2 f(t)^2$, where $f(t) = 1+13t^2+26t^4+20t^6+13t^8+22t^{10}+40t^{12}+33t^{14}+25t^{16}+33t^{18}+40t^{20}+22t^{22}+13t^{24}+20t^{26}+26t^{28}+13t^{30}+t^{32}$.

Example 9.3. $N(q,p)$-representations. $\widetilde{\nu} : G(K(r)) \longrightarrow N(q,p) \longrightarrow GL(2pq,\mathbb{Z})$.

(I) Let $q=4, p=3, N(4,3) = \mathbb{Z}/8 \otimes \mathbb{Z}/3$.

(a) $r = 1/3$. $\widetilde{\Delta}_{\widetilde{\nu},K(1/3)}(t) = (1-t^8)(1+t^8+t^{16})$.

(b) $r = 1/9$. $\widetilde{\Delta}_{\widetilde{\nu},K(1/9)} = (1-t^8)(1+t^8+t^{16})(1+t^{24}+t^{48})^3$.

(c) $r = 5/27$.

$$\begin{aligned} &\widetilde{\Delta}_{\widetilde{\nu},K(5/27)} \\ &\quad = (1-t^8)(1+t^8+t^{16})(16+31t^8+16t^{16})^2(1-79t^8+129t^{16}-79t^{24}+t^{32})^2. \end{aligned}$$

(II) Let $q=5, p=3, N(5,3) = \mathbb{Z}/10 \otimes \mathbb{Z}/3$.

(a) $r = 1/3$. $\widetilde{\Delta}_{\widetilde{\nu},K(1/3)}(t) = (1-t^{10})(1+t^{10}+t^{20})$.

(b) $r = 1/9$. $\widetilde{\Delta}_{\widetilde{\nu},K(1/9)} = (1-t^{10})(1+t^{10}+t^{20})(1+t^{30}+t^{60})^3$.

(c) $r = 5/27$.

$$\begin{aligned}&\widetilde{\Delta}_{\widetilde{\nu},K(5/27)}\\ &\quad = (1-t^{10})(1+t^{10}+t^{20})(1-228t^{10}-314t^{20}-228t^{30}+t^{40})^2\\ &\quad\ \times (1024+1201t^{20}+1024t^{40}).\end{aligned}$$

(III) Let $q = 3, p = 5$, $N(3,5) = \mathbb{Z}/6 \circledS \mathbb{Z}/5$

(a) $r = 1/5$. $\widetilde{\Delta}_{\widetilde{\nu},K(1/5)}(t) = (1-t^6)^3(1+t^6+t^{12}+t^{18}+t^{24})^3$.

(b) $r = 19/85$.

$$\begin{aligned}&\widetilde{\Delta}_{\widetilde{\nu},K(19/85)}\\ &\quad = (1-t^6)^3(1+t^6+t^{12}+t^{18}+t^{24})^3\\ &\quad\ \times (64+64t^6+48t^{12}+12t^{18}+49t^{24}+12t^{30}+48t^{36}+64t^{42}+64t^{48})\\ &\quad\ \times (1-1243t^6+3335t^{12}+1570t^{18}-2423t^{24}+6320t^{30}-992t^{36}\\ &\qquad -2181t^{42}+9451t^{48}-2181t^{54}-992t^{60}+6320t^{66}-2423t^{72}\\ &\qquad +1570t^{78}+3335t^{84}-1243t^{90}+t^{96})^2\end{aligned}$$

10. *K*-metacyclic representations

In this section, we briefly discuss K-metacyclic representations of the knot group. Let p be an odd prime. Consider a group $G(p-1,p|k)$ that has the following presentation:

$$G(p-1,p|k) = \langle s, a | s^{p-1} = a^p = 1, sas^{-1} = a^k \rangle, \tag{10.1}$$

where k is a primitive $(p-1)$-st root of 1 (mod p).

We call $G(p-1,p|k)$ a K-metacyclic group according to [3].

Proposition 10.1. *Two K-metacyclic groups of the same order, $p(p-1)$ say, are isomorphic.*

Proof. Let $G(p-1,p|\ell) = \langle u, b | u^{p-1} = b^p = 1, ubu^{-1} = b^\ell \rangle$ be another K-metacyclic group. Since ℓ is also a primitive $(p-1)$-st root (mod p), we see that $\ell \equiv k^m$ (mod p), $1 \le m \le p-2$ for some m, where m and $p-1$ are coprime. Take two integers λ and μ such that $m\lambda + (p-1)\mu = 1$. Then it is easy to show that a homomorphism $h: \ G(p-1,p|k) \to G(p-1,p|\ell)$ defined by $h(s) = u^\lambda$ and $h(a) = b$ is in fact an isomorphism. □

The following proposition is also well-known.

Proposition 10.2. *[3][5] Let p be an odd prime. Suppose that k is a primitive $(p-1)$-st root of 1 (mod p). Then the knot group $G(K)$ is mapped onto $G(p-1,p|k)$ if and only if $\Delta_K(k) \equiv 0$ (mod p).*

As is shown in [3], $G(p-1,p|k)$ is faithfully represented in S_p by

$$\sigma(a) = (123\cdots p) \text{ and } \sigma(s) = (k^{p-1}k^{p-2}\cdots k^2k). \tag{10.2}$$

Let $\pi_* :\ G(p-1,p|k) \to GL(p,\mathbb{Z})$ be a matrix representation of $G(p-1,p|k)$ via σ. Now, let $K(r)$ be a 2-bridge knot. Suppose that $\Delta_K(k) \equiv 0 \pmod p$ for some primitive $(p-1)$-st root of 1 (mod p). Then a homomorphism $\delta :\ G(K(r)) \to G(p-1,p|k)$ given by

$$\delta(x) = s \text{ and } \delta(y) = sa, \tag{10.3}$$

induces a K-metacyclic representation $\Theta = \delta \circ \pi_* :\ G(K(r)) \to GL(p,\mathbb{Z})$.

Then Conjecture A states that

$$\widetilde{\Delta}_{\Theta,K(r)}(t) = \left[\frac{\Delta_{K(r)}(t)}{1-t}\right] F(t^{p-1}). \tag{10.4}$$

We will see that (10,4) holds for the following knots including a non-2-bridge knot.

Example 10.3. (1) Consider a trefoil knot K. Since $\Delta_K(-2) \equiv 0 \pmod 7$ and -2 is a primitive 6th root of 1 (mod 7), $G(K)$ is mapped onto $G(6,7|-2)$. Then $(\delta\circ\sigma)(x) = \sigma(s) = (132645)$ and $(\delta\circ\sigma)(y) = \sigma(sa) = (146527)$ and we see $\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^6)$.

(2) Let $K = K(1/9)$. Since $K(1/9) \in H(3)$, $G(K(1/9))$ is mapped onto $G(6,7|-2)$, and

$\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^6)(1-t^6+t^{12})$.

(3) Let $K = K(5/27) \in H(3)$. Then

$\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^6)(1-7t^6+9t^{12}-7t^{18}+t^{24})$.

Example 10.4. Consider a knot $K = K(5/9)$. Since $\Delta_K(t) = 2-5t+2t^2$,

$\Delta_K(2) = 0$ and hence $G(K(5/9))$ is mapped onto $G(m,p|2)$ for any odd prime p, where m is a divisor of $p-1$.

If $p = 5$ or 11, then 2 is a primitive $(p-1)$-st root of 1 (mod p). We see then:

(i) For $p = 5$, $\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^4)$.

(ii) For $p = 11$, $\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^{10})$.

It is quite likely that we have $\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^{p-1})$, for any odd prime p such that 2 is a primitive $(p-1)$-st root of 1 (mod p).

(iii) If $p = 7$,then 2 is a primitive third root of 1 (mod 7) and hence $G(K)$ has a representation $\Theta :\ G(K) \to G(3,7|2) \to GL(7,\mathbb{Z})$ and we obtain

$\widetilde{\Delta}_{\Theta,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right](1-t^3)^2$.

Example 10.5. Consider a non-2-bridge knot $K = 8_5$ in Reidemeister-Rolfsen table. We have a Wirtinger presentation $G(K) = \langle x,y,z|R_1,R_2\rangle$, where

$$R_1 = (x^{-1}y^{-1}zyxy^{-1}x^{-1}y^{-1})x(yxyx^{-1}y^{-1}z^{-1}yx)y^{-1} \text{ and}$$
$$R_2 = (yx^{-1}y^{-1}z^{-1}x^{-1})y(xzyxy^{-1})z^{-1}. \tag{10.5}$$

Since $\Delta_K(t) = (1-t+t^2)(1-2t+t^2-2t^3+t^4)$, it follows that $\Delta_K(-1) \equiv 0 \pmod 3$ and $\Delta_K(-1) \equiv 0 \pmod 7$, and further $\Delta_K(-2) \equiv 0 \pmod 7$. Therefore, $G(K)$ is mapped onto each of the following groups: $D_3, D_7, N(2,3), N(2,7)$ and $G(6,7|-2)$, since -2 is a primitive 6-th root of 1 (mod 7).

Now we have five representations and computed their twisted Alexander polynomials.

(1) For $\rho_1 : \ G(K) \to D_3 \to GL(3,\mathbb{Z})$, defined by $\rho_1(x) = \rho_1(z) = \pi\rho(x)$ and $\rho_1(y) = \pi\rho(y)$,we have

$\widetilde{\Delta}_{\rho_1,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right] f_1(t)f_1(-t)$, where $f_1(t) = (1+t)(1+t-2t^2+t^3+t^4)$.

(2) For $\rho_2 : \ G(K) \to D_7 \to GL(7,\mathbb{Z})$, defined by $\rho_2(x) = \rho_2(y) = \pi\rho(x)$ and $\rho_2(z) = \pi\rho(y)$, we have

$\widetilde{\Delta}_{\rho_2,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right] f_2(t)f_2(-t)$, where $f_2(t) = (1+t)^3(1+2t-7t^3-13t^4-13t^5-11t^6-13t^7-13t^8-7t^9+2t^{11}+t^{12})$.

(3) For $\rho_3 : \ G(K) \to N(2,3) \to GL(12,\mathbb{Z})$, defined by $\rho_3(x) = \rho_3(z) = \widetilde{\nu}(x)$ and $\rho_3(y) = \widetilde{\nu}(y)$, we have $\widetilde{\Delta}_{\rho_3,K}(t) = (1+t^2)^2(1+5t^2+4t^4+5t^6+t^8)^2$.

(4) For $\rho_4 : \ G(K) \to N(2,7) \to GL(28,\mathbb{Z})$, defined by $\rho_4(x) = \rho_4(y) = \widetilde{\nu}(x)$ and $\rho_4(z) = \widetilde{\nu}(y)$, we have $\widetilde{\Delta}_{\rho_4,K}(t) = (1+t^2)^6(1+4t^2+2t^4+19t^6+13t^8+37t^{10}+17t^{12}+37t^{14}+13t^{16}+19t^{18}+2t^{20}+4t^{22}+t^{24})^2$.

(5) For $\rho_5 : \ G(K) \to G(6,7|-2) \to GL(7,\mathbb{Z})$, defined by $\rho_5(x) = \rho_5(z) = \Theta(x)$ and $\rho_5(y) = \Theta(y)$, we have $\widetilde{\Delta}_{\rho_5,K}(t) = \left[\frac{\Delta_K(t)}{1-t}\right] F(t)$, where $F(t) = (1-t^6)(1-72t^6-82t^{12}-72t^{18}+t^{24})$.

We note that this example also supports Conjecture A.

11. Appendix

11.1. Proof of Proposition 2.1.

Let $\theta_n(z) = c_0^{(n)} + c_1^{(n)}z + \cdots + c_n^{(n)}z^n$ be the polynomial defined in Section 2. Here $c_k^{(n)} = \binom{n+k}{2k} + 2\binom{n+k}{2k+1}$. Now we define four $n \times n$ integer matrices A, A^*, B, B^* as follows: $A = [A_{i,j}]$, where $A_{i,j} = a_{i,n-j+1}$, $A^* = [A^*_{i,j}]$, where $A^*_{i,j} = -a_{i,j-1}$, $B = [B_{i,j}]$, where $B_{i,j} = b_{i,n-j+1}$, and $B^* = [B^*_{i,j}]$, where $B^*_{i,j} = b_{i,j}$.

Here $a_{j,k}$ and $b_{j,k}$ are given as follows.

(1) $a_{j,j} = b_{j,j} = 1$ for $1 \leq j \leq n$.

(2) For $1 \leq j \leq k, a_{j,k} = \binom{j+k-1}{2j-1}$ and $b_{j,k} = \binom{j+k-2}{2j-2}$.

(3) If $0 \leq k < j, a_{j,k} = b_{j,k} = 0$. (11.1)

Lemma 11.1. *The following formulas hold.*

For $0 \leq k \leq n$,

(1) $c_k^{(n)} = a_{k+1,n+1} + a_{k+1,n}$.

For $1 \leq j \leq k$,

(2) $b_{j,k} = a_{j,k} - a_{j,k-1}$,

(3) $b_{j,k} = a_{j-1,k-1} + b_{j,k-1}$ *and*

$$(4)\quad -2\sum_{k=j}^{n} b_{j,k} = a_{j-1,n} - c_{j-1}^{(n)} + b_{j,n}. \tag{11.2}$$

Proof. Only (4) needs a proof. Since $\sum_{k=j}^{n} b_{j,k} = \sum_{k=j}^{n}(a_{j,k} - a_{j,k-1}) = a_{j,n}$, we need to show that $-2a_{j,n} = a_{j-1,n} - c_{j-1}^{(n)} + b_{j,n}$. However, it follows easily from (11.2) (1)-(3). □

Now these formulas are sufficient to show that the $2n \times 2n$ matrix $U_n = \begin{bmatrix} A & A^* \\ B & B^* \end{bmatrix}$ is what we sought. Since a proof is straightforward, we omit the details.

Example 11.2. For $n = 4, 5$, U_n are given by

$$U_4 = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ 10 & 4 & 1 & 0 & 0 & 0 & -1 & -4 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 3 & 1 & 0 & 0 & 1 & 3 & 6 \\ 5 & 1 & 0 & 0 & 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } U_5 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\ 20 & 10 & 4 & 1 & 0 & 0 & 0 & -1 & -4 & -10 \\ 21 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -6 \\ 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 6 & 3 & 1 & 0 & 0 & 1 & 3 & 6 & 10 \\ 15 & 5 & 1 & 0 & 0 & 0 & 0 & 1 & 5 & 15 \\ 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

11.2. Proof of Lemma 5.2.

First we write down a solution $X = V_n$ of the equation $X^2 = 4E_n + C_n$. Let us begin with the alternating Catalan series

$$\mu(y) = \sum_{k=0}^{\infty} b_k y^k, \text{ where } b_k = \frac{(-1)^k}{k+2}\binom{2k+2}{k+1}. \tag{11.3}$$

Therefore, $\mu(y) = 1 - 2y + 5y^2 - 14y^3 + 132y^4 - 429y^5 + 1430y^6 + \cdots$. Let $\theta_n(z) = c_0^{(n)} + c_1^{(n)}z + \cdots + c_n^{(n)}z^n$ be the polynomial defined in Section 2. Using $\theta_n(z)$, we define a new polynomial $f_n(x) = x^n\theta(x^{-1}) = a_0^{(n)} + a_1^{(n)}x + a_2^{(n)}x^2 + \cdots + a_n^{(n)}x^n$. For example, $f_1(x) = x\theta_1(x^{-1}) = x(3 + x^{-1}) = 3x + 1$, and $f_2(x) = 5x^2 + 5x + 1$. Since $a_k^{(n)} = c_{n-k}^{(n)}$, we see that

$$a_k^{(n)} = \frac{2n+1}{2n-2k+1}\binom{2n-k}{2n-2k} = \binom{2n-k+1}{2n-2k+1} + \binom{2n-k}{2n-2k+1}. \tag{11.4}$$

Next, we compute $f_n(x)\mu(y) = \sum_{r,s\geq 0} c_{r,s}^{(n)} x^r y^s$, where $c_{r,s}^{(n)} = a_r^{(n)} b_s$, and define integers $d_{k,\ell}^{(n)}$, $0 \leq k, \ell$, as follows:

$$d_{k,\ell}^{(n)} = c_{k,\ell}^{(n)} + c_{k-1,\ell+1}^{(n)} + c_{k-2,\ell+2}^{(n)} + \cdots + c_{0,k+\ell}^{(n)} = \sum_{i=0,i+j=k+\ell}^{k} a_i^{(n)} b_j. \tag{11.5}$$

Then we claim:

Proposition 11.3. *$V_n = [v_{j,k}^{(n)}]_{1\leq j,k\leq n}$, where $v_{j,k}^{(n)} = d_{n-j,k-1}^{(n)}$, is a solution.*

Example 11.4. The following is the list of solutions $V_n, n = 1, \ldots, 5$.

$$[1], \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 5 & -7 & 14 \\ 5 & -9 & 21 \\ 1 & -2 & 5 \end{bmatrix}, \begin{bmatrix} 7 & -9 & 18 & -45 \\ 14 & -23 & 51 & -132 \\ 7 & -13 & 31 & -84 \\ 1 & -2 & 5 & -14 \end{bmatrix}, \begin{bmatrix} 9 & -11 & 22 & -55 & 154 \\ 30 & -46 & 99 & -253 & 715 \\ 27 & -47 & 108 & -286 & 825 \\ 9 & -17 & 41 & -112 & 330 \\ 1 & -2 & 5 & -14 & 42 \end{bmatrix}.$$

Now, to prove Proposition 11.3, we need several technical lemmas.

Lemma 11.5. *For $n \geq 2$ and $0 \leq k \leq n$, the following recursion formula holds.*

$$a_k^{(n)} = a_k^{(n-1)} + 2a_{k-1}^{(n-1)} - a_{k-2}^{(n-2)}. \tag{11.6}$$

For convenience, we define $a_0^{(0)} = 1$. Since a direct computation using (11.4) verifies (11.6) easily, we omit details.

Next, for $n, m \geq 0$, we define a number $F(n,m)$ as follows.

$$F(n,m) = \sum_{j=0}^{n} a_{n-j}^{(n)} b_{m+j}. \tag{11.7}$$

Example 11.6. We have the following values for $F(n,m)$;

(1) (*i*) $F(0,0) = a_0^{(0)} b_0 = 1$.

(*ii*) $F(0,m) = a_0^{(n)} b_m = b_m$.

(2) (*i*) $F(1,0) = a_1^{(1)} b_0 + a_0^{(1)} b_1 = 3 - 2 = 1$.

(*ii*) $F(1,1) = a_1^{(1)} b_1 + a_0^{(1)} b_2 = -6 + 5 = -1$.

(*iii*) $F(1,m) = a_1^{(1)} b_m + a_0^{(1)} b_{m+1} = 3b_m + b_{m+1}$.

(3) (*i*) $F(2,0) = a_2^{(2)} b_0 + a_1^{(2)} b_1 + a_0^{(2)} b_2 = 0$.

(*ii*) $F(2,1) = 1$.

(*iii*) $F(2,2) = -3$.

Lemma 11.7. *For $n \geq 2$ and $m \geq 0$, the following recursion formula holds.*

$$F(n,m) = F(n-1,m+1) + 2F(n-1,m) - F(n-2,m). \tag{11.8}$$

Proof. Use (11.6) to show (11.8) as follows:

$$\begin{aligned} F(n,m) &= \sum_{j=0}^{n} a_{n-j}^{(n)} b_{m+j} = \sum_{j=0}^{n} [a_{n-j}^{(n-1)} + 2a_{n-1-j}^{(n-1)} - a_{n-2-j}^{(n-2)}] b_{m+j} \\ &= \sum_{j=0}^{n-1} a_{n-1-j}^{(n-1)} b_{m+1+j} + 2\sum_{j=0}^{n-1} a_{n-1-j}^{(n-1)} b_{m+j} - \sum_{j=0}^{n-2} a_{n-2-j}^{(n-2)} b_{m+j} \\ &= F(n-1,m+1) + 2F(n-1,m) - F(n-2,m). \qquad \square \end{aligned}$$

Lemma 11.8. *The following formulas hold.*

(1) *For* $n \geq 1$ *and* $0 \leq k \leq n, \sum_{j=0}^{n} a_{k-j}^{(k)} b_j = a_k^{(n-1)}$.

(2) *For* $n \geq 2$ *and* $0 \leq m \leq n-2, F(n,m) = 0$.

(3) *For* $n \geq 1, F(n,n-1) = 1$.

(4) *For* $n \geq 1, F(n,n) = -(2n-1)$. (11.9)

Proof. (1) Use induction on n. Since (1) holds for $n = 1$, we may assume that it holds for n. Further, if $k = 0$, (1) holds trivially, and hence it suffices to show (1) for $n = n+1$ and $k = k+1$. Then, by (11.6),

$$\begin{aligned} \sum_{j=0}^{k+1} a_{k+1-j}^{(n+1)} b_j &= \sum_{j=0}^{k+1} \{a_{k+1-j}^{(n)} + 2a_{k-j}^{(n)} - a_{k-1-j}^{(n-1)}\} b_j \\ &= \sum_{j=0}^{k+1} a_{k+1-j}^{(n)} b_j + 2\sum_{j=0}^{n} a_{k-j}^{(n)} b_j - \sum_{j=0}^{k-1} a_{k-1-j}^{(n-1)} b_j \\ &= a_{k+1}^{(n-1)} + 2a_k^{(n-1)} - a_{k-1}^{(n-2)} \\ &= a_{k+1}^{(n)}. \end{aligned}$$

Proof of (2). Since $F(n,m+1) = F(n+1,m) - 2F(n,m) + F(n-1,m)$, it suffices to show that $F(n,0) = 0$ if $n \geq 2$.

Now

$$\begin{aligned} F(n,0) &= \sum_{j=0}^{n} a_{n-j}^{(n)} b_j = \sum_{j=0}^{n} \frac{(2n+1)}{(2j+1)} \binom{n+j}{2j} \frac{(-1)^j}{(j+2)} \binom{2j+2}{j+1} \\ &= (2n+1) \sum_{j=0}^{n} (-1)^j \frac{(n+j)!}{(2j+1)!(n-j)!} \frac{(2j+2)!}{(j+2)!(j+1)!} \\ &= (2n+1) \sum_{j=0}^{n} (-1)^j \frac{(n+j)!(2j+2)}{(n-j)!(j+2)!(j+1)!} \\ &= (2n+1) \sum_{j=0}^{n} (-1)^j \frac{2(n+j)!}{(n-j)!(j+2)!j!}. \end{aligned}$$

Therefore, to prove (2), it suffices to show

$$\sum_{j=0}^{n}(-1)^j \frac{(n+j)!}{(n-j)!(j+2)!j!} = 0 \tag{11.10}$$

or equivalently, by multiplying both sides through $n!/(n-2)!$, to show

$$\sum_{j=0}^{n}(-1)^j \binom{n}{j}\binom{n+j}{j+2} = 0. \tag{11.11}$$

To show (11.11), we apply the following lemma [8, Lemma 5.3].

Lemma 11.9. *For $N \geq M \geq 0$ and $N \geq K \geq 0$,*

$$\begin{aligned}&\binom{N}{K}\binom{M}{M} - \binom{N-1}{K-1}\binom{M}{M-1} + \binom{N-2}{K-2}\binom{M}{M-2} - \cdots \\ &+ (-1)^M \binom{N-M}{K-M}\binom{M}{0} \\ &= \binom{N-M}{K}.\end{aligned} \tag{11.12}$$

Put $N = 2n$, $K = n+2$ and $M = n$ in (11.12). Since $N - M = n < K$, we see

$$\binom{2n}{n+2}\binom{n}{n} - \binom{2n-1}{n+1}\binom{n}{n-1} + \cdots + (-1)^n \binom{n}{2}\binom{n}{0} = \binom{n}{n+2} = 0,$$

and hence $\sum_{j=0}^{n}(-1)^j \binom{n+j}{j+2}\binom{n}{j} = 0$. This proves (2).

Proof of (3). By (11.8), we see that for $n \geq 2$,

$$F(n+1, n-2) = F(n, n-1) + 2F(n, n-2) - F(n-1, n-2).$$

Since $F(n+1, n-2) = F(n, n-2) = 0$ by (11.9) (2), it follows that $F(n, n-1) = F(n-1, n-2)$, and hence $F(n, n-1) = F(1, 0) = 1$ by Example 11.6 (2)(i).

Proof of (4). Use (11.8) for $n \geq 1$ to see

$$F(n+1, n-1) = F(n, n) + 2F(n, n-1) - F(n-1, n-1).$$

Since $F(n+1, n-1) = 0$ and $F(n, n-1) = 1$, it follows that

$$F(n, n) = F(n-1, n-1) - 2$$

and hence,

$$F(n, n) = F(1, 1) - 2(n-1) = -1 - 2n + 2 = -(2n-1).$$

$\square$

We define another number $H_k^{(n)}$ as follows. For any $n \geq 1$ and $k \geq 2$, we define

$$H_k^{(n)} = \sum_{j=0}^{k} a_j^{(n)} F(n-1, n+k-2-j) - \sum_{j=0}^{k-2} a_j^{(n-1)} F(n, n+k-3-j). \tag{11.13}$$

For example, $H_2^{(5)} = a_0^{(5)}F(4,5) + a_1^{(5)}F(4,4) + a_2^{(5)}F(4,3) - a_0^{(4)}F(5,4) = 0$.
In particular, we should note;

$$\text{For } k \geq 2, H_k^{(1)} = 0. \tag{11.14}$$

$$\begin{aligned}\text{In fact, } H_k^{(1)} &= a_0^{(1)}F(0,k-1) + a_1^{(1)}F(0,k-2) - a_0^{(0)}F(1,k-2)\\ &= b_{k-1} + a_1^{(1)}b_{k-2} - 3b_{k-2} - b_{k-1}\\ &= 0.\end{aligned}$$

The last formula we need is the following lemma.

Lemma 11.10. *For any $n \geq 1$ and $k \geq 2$, we have $H_k^{(n)} = 0$.* *(11.15)*

Proof. We compute $H = H_k^{(n)} - H_k^{(n-1)}$. By definition, for $n \geq 2$,

$$\begin{aligned}H = &-\sum_{j=0}^{k-2} a_j^{(n-1)}F(n, n+k-3-j) + a_0^{(n)}F(n-1, n+k-2)\\ &+ a_1^{(n)}F(n-1, n+k-3) + \sum_{j=0}^{k-2}(a_{j+2}^{(n)} + a_j^{(n-2)})F(n-1, n+k-4-j)\\ &- \sum_{j=0}^{k} a_j^{(n-1)}F(n-2, n+k-3-j).\end{aligned}$$

Since $a_{j+2}^{(n)} + a_j^{(n-2)} = a_{j+2}^{(n-1)} + 2a_{j+1}^{(n-1)}$ and $a_1^{(n)} = a_1^{(n-1)} + 2a_0^{(n-1)}$ by (11.6), we see

$$\begin{aligned}H = &-\sum_{j=0}^{k-2} a_j^{(n-1)}F(n, n+k-3-j) + a_0^{(n)}F(n-1, n+k-2)\\ &+ (a_1^{(n-1)} + 2a_0^{(n-1)})F(n-1, n+k-3)\\ &+ \sum_{j=0}^{k-2}(a_{j+2}^{(n-1)} + 2a_{j+1}^{(n-1)})F(n-1, n+k-4-j)\\ &- \sum_{j=0}^{k} a_j^{(n-1)}F(n-2, n+k-3-j).\end{aligned}$$

Note $a_0^{(n)} = a_0^{(n-1)} = 1$ to see

$$
\begin{aligned}
H = a_0^{(n-1)}&\{-F(n,n+k-3)+F(n-1,n+k-2)\\
&\quad +2F(n-1,n+k-3)-F(n-2,n+k-3)\}\\
&+a_1^{(n-1)}\{-F(n,n+k-4)+F(n-1,n+k-3)\\
&\quad +2F(n-1,n+k-4)-F(n-2,n+k-4)\}+\cdots\\
&+a_{k-2}^{(n-1)}\{-F(n,n-1)+F(n-1,n)\\
&\quad +2F(n-1,n-1)-F(n-2,n-1)\}\\
&+a_{k-1}^{(n-1)}\{F(n-1,n-1)+2F(n-1,n-2)-F(n-2,n-2)\}\\
&+a_k^{(n-1)}\{F(n-1,n-2)-F(n-2,n-3)\}.
\end{aligned}
$$

By (11.8) and (11.9)(3),(4), we see easily that each term of the summation is equal to 0. This proves $H=0$. □

Now we are in position to prove Proposition 11.3. Let $\mathbf{u}_j = (v_{j,1}^{(n)}, v_{j,2}^{(n)}, \cdots, v_{j,n}^{(n)})$ and $\mathbf{w}_k = (v_{1,k}^{(n)}, v_{2,k}^{(n)}, \cdots, v_{n,k}^{(n)})^T$ be, respectively, the j-th row vector and the k-th column vector of V_n. Then we must show

(1) $\mathbf{u}_{n-j}\cdot\mathbf{w}_k = 0$ *for* (*i*) $0\le j\le n-3, 1\le k\le n-j-2$ *and*
(*ii*) $2\le j\le n-1, n-j+1\le k\le n-1$.

(2) $\mathbf{u}_{n-j}\cdot\mathbf{w}_{n-j-1} = 1$ *for* $0\le j\le n-2$.

(3) $\mathbf{u}_{n-j}\cdot\mathbf{w}_{n-j} = 4,$ *for* $1\le j\le n-1$.

(4) $\mathbf{u}_n\cdot\mathbf{w}_n = 4-a_1^{(n)} = 4-c_{n-1}^{(n)}$,

(5) $\mathbf{u}_{n-j}\cdot\mathbf{w}_n = -a_{j+1}^{(n)} = -c_{n-j-1}^{(n)}$ *for* $1\le j\le n-1$. (11.16)

Since (11.16) is obviously true for $n=1$, we assume hereafter that $n\ge 2$.

We introduce new vectors, $\mathbf{b}_j = (b_j, b_{j+1}, \cdots, b_{j+n-1})$ for $j\ge 0$ and $\mathbf{a}_k^{(n)} = (a_k^{(n)}, a_{k-1}^{(n)}, \cdots, a_0^{(n)}, 0, \cdots, 0)^T$ for $0\le k\le n$. Then, from the definition of $v_{j,k}^{(n)}$, it is easy to see the following:

(1) For $0\le j\le n-1, \mathbf{u}_{n-j} = a_j^{(n)}\mathbf{b}_0 + a_{j-1}^{(n)}\mathbf{b}_1+\cdots+a_0^{(n)}\mathbf{b}_j$.

(2) For $1\le k\le n, \mathbf{w}_k = b_{k-1}\mathbf{a}_{n-1}^{(n)} + b_k\mathbf{a}_{n-2}^{(n)}+\cdots+b_{n+k-2}\mathbf{a}_0^{(n)}$. (11.17)

Since $\mathbf{u}_{n-j}\cdot\mathbf{w}_k = \sum_{i=0}^{j} a_{j-i}^{(n)}(\mathbf{b}_i\cdot\mathbf{w}_k)$, we first compute $\mathbf{b}_i\cdot\mathbf{w}_k$. In fact, a straightforward computation shows

$$
\begin{aligned}
\mathbf{b}_i\cdot\mathbf{w}_k &= b_{k-1}(a_{n-1}^{(n)}b_i + a_{n-2}^{(n)}b_{i+1}+\cdots+a_0^{(n)}b_{n+i-1})\\
&\quad + b_k(a_{n-2}^{(n)}b_i + a_{n-3}^{(n)}b_{i+1}+\cdots+a_0^{(n)}b_{n+i-2})+\cdots+b_{n+k-2}(a_0^{(n)}b_i)\\
&= b_{k-1}(F(n,i-1)-a_n^{(n)}b_{i-1})\\
&\quad + b_k(F(n,i-2)-a_{n-1}^{(n)}b_{i-1}-a_n^{(n)}b_{i-2})\\
&\quad +\cdots\\
&\quad + b_{k+i-2}(F(n,0)-a_{n-i+1}^{(n)}b_{i-1}-\cdots-a_n^{(n)}b_0)
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{k+i-1}(a_{n-1}^{(n-1)} - a_{n-i}^{(n)} b_{i-1} - \cdots - a_{n-1}^{(n)} b_0)\\
&+ \cdots\\
&+ b_{n+k-2}(a_i^{(n-1)} - a_1^{(n)} b_{i-1} - a_2^{(n)} b_{i-2} - \cdots - a_i^{(n)} b_0)\\
&+ b_{n+k-1}(a_{i-1}^{(n-1)} - a_0^{(n)} b_{i-1} - a_1^{(n)} b_{i-2} - \cdots - a_{i-1}^{(n)} b_0)\\
&+ \cdots\\
&+ b_{n+k+i-2}(a_0^{(n-1)} - a_0^{(n)} b_0).
\end{aligned}
$$

Note that in the above summation, each of the last i terms is 0 by (11.9)(1).

By rearranging this summation, we obtain

$$
\begin{aligned}
\mathbf{b}_i \cdot \mathbf{w}_k = {} & b_{k-1} F(n, i-1) + b_k F(n, i-2) + \cdots\\
&+ b_{k+i-2} F(n, 0) + F(n-1, k+i-1)\\
&- b_{i-1} F(n, k-1) - b_{i-2} F(n, k) - \cdots - b_0 F(n, k+i-2).
\end{aligned}
$$

Since $0 \le i \le j \le n-1$, we have for $\ell \ge 0$, $i-1-\ell \le n-2$ and hence $F(n, i-1-\ell) = 0$. Therefore

$$
\mathbf{b}_i \cdot \mathbf{w}_k = F(n-1, k+i-1) - \sum_{\ell=0}^{i-1} b_{i-1-\ell} F(n, k-1+\ell). \tag{11.18}
$$

Case 1. $i = 0$. Then $\mathbf{b}_0 \cdot \mathbf{w}_k = F(n-1, k-1)$. If $1 \le k \le n-2$, then $F(n-1, k-1) = 0$, and hence $\mathbf{u}_n \cdot \mathbf{w}_k = a_0^{(n)}(\mathbf{b}_0 \cdot \mathbf{w}_k) = 0$. Further,

$$
\begin{aligned}
&\mathbf{u}_n \cdot \mathbf{w}_{n-1} = a_0^{(n)} F(n-1, n-2) = a_0^{(n)} = 1, \text{ and}\\
&\mathbf{u}_n \cdot \mathbf{w}_n = a_0^{(n)} F(n-1, n-1) = -(2n-3) = 4 - (2n+1) = 4 - a_1^{(n)}.
\end{aligned}
$$

This proves (11.16) for $j = 0$.

Case 2. $i = 1$. Then $\mathbf{b}_1 \cdot \mathbf{w}_k = F(n-1, k) - b_0 F(n, k-1)$. If $1 \le k \le n-3$, then $F(n-1, k) = F(n, k-1) = 0$ and $\mathbf{b}_1 \cdot \mathbf{w}_k = 0$. Since $\mathbf{b}_0 \cdot \mathbf{w}_k = 0$, we have $\mathbf{u}_{n-1} \cdot \mathbf{w}_k = 0$ for $1 \le k \le n-3$. Further,

$$
\begin{aligned}
\mathbf{u}_{n-1} \cdot \mathbf{w}_{n-2} &= a_1^{(n)}(\mathbf{b}_0 \cdot \mathbf{w}_{n-2}) + a_0^{(n)}(\mathbf{b}_1 \cdot \mathbf{w}_{n-2})\\
&= a_1^{(n)} F(n-1, n-3) + a_0^{(n)}\{F(n-1, n-2) - b_0 F(n, n-3)\}\\
&= a_0^{(n)} F(n-1, n-2) = 1.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathbf{u}_{n-1} \cdot \mathbf{w}_{n-1} &= a_1^{(n)} F(n-1, n-2) + a_0^{(n)}\{F(n-1, n-1) - b_0 F(n, n-2)\}\\
&= a_1^{(n)} + a_0^{(n)}(-(2n-3)) = 2n+1-(2n-3) = 4.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbf{u}_{n-1} \cdot \mathbf{w}_n &= a_1^{(n)} F(n-1, n-1) + a_0^{(n)}\{F(n-1, n) - b_0 F(n, n-1)\}\\
&= H_2^{(n)} - a_2^{(n)} = -a_2^{(n)}, \text{ by (11.15).}
\end{aligned}
$$

This proves (11.16) for $j = 1$.

Now we assume that $2 \leq j \leq n-1$ and compute $\mathbf{u}_{n-j} \cdot \mathbf{w}_k$, $1 \leq k \leq n$. Then

$$\begin{aligned}
\mathbf{u}_{n-j} \cdot \mathbf{w}_k &= \sum_{i=0}^{j} a_{j-i}^{(n)} (\mathbf{b}_i \cdot \mathbf{w}_k) \\
&= a_j^{(n)} (\mathbf{b}_0 \cdot \mathbf{w}_k) + \sum_{i=1}^{j} a_{j-i}^{(n)} (\mathbf{b}_i \cdot \mathbf{w}_k) \\
&= a_j^{(n)} F(n-1, k-1) + \sum_{i=1}^{j} a_{j-i}^{(n)} \{F(n-1, k+i-1) \\
&\quad - \sum_{\ell=0}^{i-1} b_{i-1-\ell} F(n, k-1+\ell)\} \\
&= \sum_{i=0}^{j} a_{j-i}^{(n)} F(n-1, k+i-1) - \sum_{i=1}^{j} a_{j-i}^{(n)} \sum_{\ell=0}^{i-1} b_{i-1-\ell} F(n, k-1+\ell).
\end{aligned}$$

Therefore, the coefficient of $F(n, k-1+q)$, $0 \leq q \leq i-1$, is equal to

$$\begin{aligned}
\sum_{i=1}^{j} a_{j-i}^{(n)} b_{i-1-q} &= \sum_{i=q+1}^{j} a_{j-i}^{(n)} b_{i-1-q} \\
&= a_{j-q-1}^{(n-1)} \text{ by (11.9)(1), and hence}
\end{aligned}$$

$$\mathbf{u}_{n-j} \cdot \mathbf{w}_k = \sum_{i=0}^{j} a_{j-i}^{(n)} F(n-1, k+i-1) - \sum_{q=0}^{j-1} a_{j-q-1}^{(n-1)} F(n, k-1+q). \tag{11.19}$$

If $1 \leq k \leq n-j-2$, then $k+i-1 \leq k+j-1 \leq n-3$ and also $k-1+q \leq k+j-2 \leq n-4$, and hence, $\mathbf{u}_{n-j} \cdot \mathbf{w}_k = 0$.

If $k = n-j-1$, then $\mathbf{u}_{n-j} \cdot \mathbf{w}_{n-j-1} = a_0^{(n)} F(n-1, n-2) = a_0^{(n)} = 1$. Further, $\mathbf{u}_{n-j} \cdot \mathbf{w}_{n-j} = a_0^{(n)} F(n-1, n-1) + a_1^{(n)} F(n-1, n-2) = -(2n-3) + 2n + 1 = 4$.

Now suppose $n-j+1 \leq k \leq n-1$. Then by (11.9),

$$\begin{aligned}
\mathbf{u}_{n-j} \cdot \mathbf{w}_k &= \sum_{i=0}^{j} a_{j-i}^{(n)} F(n-1, k+i-1) - \sum_{q=0}^{j-1} a_{j-q-1}^{(n-1)} F(n, k-1+q) \\
&= \sum_{i=n-k-1}^{j} a_{j-i}^{(n)} F(n-1, k+i-1) - \sum_{q=n-k}^{j-1} a_{j-q-1}^{(n-1)} F(n, k-1+q).
\end{aligned}$$

That is exactly $H_{k-(n-j-1)}^{(n)}$ and hence $\mathbf{u}_{n-j} \cdot \mathbf{w}_k = 0$ for $n-j+1 \leq k \leq n-1$.

Finally, a similar computation shows that

$$\begin{aligned}\mathbf{u}_{n-j}\cdot\mathbf{w}_n &= \sum_{i=0}^{j} a_{j-i}^{(n)} F(n-1,n+i-1) - \sum_{q=0}^{j-1} a_{j-1-q}^{(n-1)} F(n,n+q-1)\\ &= H_{j+1}^{(n)} - a_{j+1}^{(n)} F(n-1,n-2)\\ &= -a_{j+1}^{(n)}\\ &= -c_{n-j-1}^{(n)}.\end{aligned}$$

This proves (11.16) and a proof of Proposition 11.3 is now complete. □

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